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PRIME IDEALS IN HOPF GALOIS EXTENSIONS

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ABSTRACT

For a finite-dimensional Hopf algebra H, we study the prime ideals in a faithfully flat H-Hopf-Galois extension $R \subset A$. One application is to quotients of Hopf algebras which arise in the theory of quantum groups at a root of 1. For the Krull relations between R and A, we obtain our best results when H is semisolvable; these results generalize earlier known results for crossed products for a group action and for algebras graded by a finite group. We also show that if H is semisimple and semisolvable, then A is semiprime provided R is H-semiprime.

0. Introduction

Let H be a finite-dimensional Hopf algebra over a field k and $R \subset A$ a faithfully flat H-Galois extension. Intuitively, the algebra extension $R \subset A$ represents an epimorphism of quantum spaces which is a principal bundle with fibre being the quantum group corresponding to H (where the category of quantum groups is the dual category to the category of Hopf algebras). Formally, faithfully flat H-Galois extensions are defined as follows (see [M, Chapter 8]):

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Let A be a right H-comodule algebra with structure map $\rho : A \to A \otimes H$, that is an algebra map such that A is an H-comodule via ρ. Define

$$R := A^{\mathrm{co}\,H} = \{a \in A \mid \rho(a) = a \otimes 1\},$$

the algebra of H-coinvariant elements of A. Then $R \subset A$ is **faithfully flat H-Galois** if the Galois map

$$A \otimes_R A \to A \otimes H, \quad x \otimes y \mapsto x\rho(y),$$

is bijective, and A is faithfully flat as a left (or equivalently right) R-module.

The object of this paper is to compare the prime ideals of R and of A, with particular interest in the classical Krull relations. Since Hopf crossed products $A = R\#_\sigma H$ give examples of faithfully flat Galois extensions (see [M, Chapter 7]), our results apply to crossed products. Another important class of examples we have in mind are Hopf algebras A with a normal sub Hopf algebra R of finite index; if A is faithfully flat over R, then $R \subset A$ is faithfully flat H-Galois where $H := A/AR^+$ is the quotient Hopf algebra. Interesting examples of such Hopf algebra extensions occur in the theory of quantum groups when the deformation parameter is a root of unity.

For the Krull relations we obtain our best results when H has a normal series whose quotients are commutative or cocommutative; this generalizes our earlier work [MS] as well as work of Lorenz and Passman for crossed products of groups [LP] and work of Cohen and the first author for smash products over group-graded rings [CM]. As a consequence we make progress on the question of when R being H-prime and H being semisimple implies that $R\#H$ is semiprime, although the general question remains open.

A major motivation for studying Galois extensions rather than crossed products is that crossed products are not transitive, by an example of the second author [S3]. That is, if $A = R\#_\sigma H$ is a crossed product, K a normal Hopf subalgebra of H with quotient $\bar{H}$, then it is false in general that one can write $A = (R\#_\sigma K)\#_\tau \bar{H}$. As shown in Section 6, however, faithfully flat Galois extensions are transitive, and thus inductive arguments are possible. Hence to obtain results on smash products by inductive arguments it is necessary to study general faithfully flat Galois extensions. On the other hand, it turns out that usually the hard case is the case of an arbitrary smash product extension. If H is given, then some property should hold for all faithfully flat H-Galois extensions $R \subset A$ if it holds for all smash products $R \subset R\#H$.

Smash product extensions $R \subset R\#H$ are normalizing extensions when H is a group algebra, but not in general. Hence the rich theory of prime ideals in normalizing ring extensions (see [McR]) does not apply here.

OUTLINE OF THE PAPER. In Section 1 we introduce the notion of H-stable ideals of R, where $R \subset A$ is faithfully flat H-Galois. We show that such ideals behave analogously to the usual notion of H-stable ideals when there is actually an H-action. The basic tool here is the Morita equivalence between R and $A\#H^*$, where the H^*-action on A is the dual of the given H-coaction.

In Section 2 we apply the Morita equivalence to H-primes of R and H^*-primes of A, and get a bijective correspondence between H-equivalent primes of R and H^*-equivalent primes of A

$$\mathrm{Spec}(A)/\sim_{H^*} \xrightarrow{\cong} \mathrm{Spec}(R)/\sim_H .$$

As a special case (Corollary 2.6), we show that if also H^* is pointed and $R \subset A$ is a centralizing extension, then there is a bijection between Spec(R) and the orbits of Spec(A) under the action of the character group of H. The corollary is applied to the central extension $F_0[G] \subset \mathcal{O}_\varepsilon[G]$ of the quantum coordinate algebra over the usual coordinate algebra, where G is the connected, simply connected, semisimple algebraic group corresponding to a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and ε is a primitive l-th root of unity, l odd and prime to 3 in case $\mathfrak{g}$ has a G_2 component. In Corollary 2.8 we show that $F_0[G] \subset \mathcal{O}_\varepsilon[G]$ is faithfully flat H-Galois, where H is the quotient Hopf algebra, by general Hopf algebra arguments (in particular $\mathcal{O}_\varepsilon[G]$ is projective over $F_0[G]$, a fact which was shown in [DL] in a completely different way), and we derive a bijection

$$\mathrm{Spec}(\mathcal{O}_\varepsilon[G])/\chi \xrightarrow{\cong} \mathrm{Spec}(F_0[G]),$$

where χ is the character group of H. The prime correspondence in this case was shown independently by E. Letzter [L2] by different methods.

In Section 3 we show that H-Spec(R) can be identified with H_0-Spec(R), where H_0 is the coradical of H. This generalizes work of Chin [Ch90] for the case when H is pointed, that is $H_0 = kG$. For this we must introduce C-stable ideals and C-primes for a subcoalgebra C of H.

In Section 4, we define versions of the Krull relations such as going up (GU), incomparability (INC), and lying over, though we replace the usual lying over property with a related property called t-lying over (t-LO). We also define the "duals" of these three relations, which we call co-going up (co-GU), co-incomparability (co-INC), and t-colying over (t-coLO); we say that H has a given Krull relation if the relation holds for all faithfully flat H-Galois extensions $R \subset A$. We then prove that the dual relations are indeed dual, in the sense that H will have a given relation if and only if H^* has the dual relation. We also show that to see if H has a given relation, it suffices to check it for $R \subset A$ in the special case when $A = R\#H$ and R is H-prime.

In fact the H-Galois assumption can be weakened for some of our results; this is the topic of Section 5. We consider there H-module algebras A with invariant ring $R = A^H$ such that the trace map $\hat{t} : A \to R$ is surjective (in coaction language, this is equivalent to $R \subset A$ being an H^*-extension with a total integral). Our assumption implies that there exists an idempotent $0 \neq e = e^2 \in A\#H$ such that $e(A\#H)e \cong A^H$. Using this fact and some techniques from work on group actions, we are able to prove fairly general results on the Krull relations for $R \subset A$ (Theorem 5.5). As a consequence we are able to define an equivalence relation on Spec(R) and prove analogs of the correspondence discussed in Section 2. This generalizes work of [M81] for group actions.

In Section 6 we prove some very general transitivity results for the Krull relations. That is, if K is a normal sub Hopf algebra of H with quotient Hopf algebra $\bar{H}$, when do the Krull relations for K and $\bar{H}$ imply those for H? In fact we prove more general versions of transitivity, for arbitrary ring extensions with a "lying over relation". This enables us to enlarge our base field (in Section 7) as well as to look at Hopf Galois extensions. We note that the difficulties in Section 7 involve reflecting the various Krull relations from $H \otimes E$ back to H, where $k \subset E$ is a field extension.

Finally, in Section 8 we obtain consequences of the work of the previous sections. We first prove a few facts about quotients of Hopf algebras (Lemma 8.2) to enable us to apply our transitivity results to Hopf algebras with various normal series; recall that a Hopf algebra is **(co)solvable** if it has a normal series in which all of the quotients are (co)commutative. We prove (Theorem 8.4) that when $t = \dim H$ and H is cosolvable (resp. solvable), then H has t-coLO (resp. t-LO) and GU (resp. coGU). If H is both solvable and cosolvable, for example if H is the restricted universal enveloping algebra of a solvable restricted Lie algebra, then H has all six Krull relations. We note that it is an open question whether any finite-dimensional cocommutative Hopf algebra has all the Krull relations. It might even be true that any finite-dimensional Hopf algebra has all the Krull relations.

When H is semisimple, more can be shown. We say that H is **semisolvable** if H has a normal series in which the quotients are either commutative or cocommutative. If H is semisimple and semisolvable, then we prove that it has 1-LO,

t-coLO, GU, coGU, INC, and coINC (Theorem 8.5). In particular, let R be any H-module algebra and $A := R\#H$ where H is semisimple and semisolvable. Then we get from Theorem 8.5:

1. If R is H-prime, then A has at most $n \leq \dim H$ minimal primes, call them $P_1 \ldots, P_n$; $\bigcap_{i=1}^n P_i = \{0\}$; and $P \in \mathrm{Spec}(A)$ is a minimal prime if and only if $P \cap R = \{0\}$ (since H has 1-LO and INC).

2. If R is H-semiprime, then $R\#H$ is semiprime (since H has 1-LO).

In fact H is known to be semisolvable in many cases; for example, when k is algebraically closed of characteristic 0, $\dim H$ is a power of a prime and H is semisimple, then H has a normal series in which all the quotients are both commutative and cocommutative.

1. H-stable ideals of R

Throughout this section, $R \subset A$ denotes a faithfully flat right H-Galois extension. We introduce here the notion of an H-stable ideal of R; in the special case when R is an H-module algebra and $A = R\#H$, it becomes the usual notion of H-stable ideal.

We first recall a general result (see for example [S1]); for $R \subset A$ as above, and assuming H has a bijective antipode, there exist category equivalences as follows:

(i) $\mathcal{M}_R \leftrightarrows \mathcal{M}_A^H$, given by $M \mapsto M \otimes_R A \cong MA$ and $N \mapsto N^{\mathrm{co}H}$, for $M \in \mathcal{M}_R$ and $N \in \mathcal{M}_A^H$;

(ii) ${}_R\mathcal{M} \leftrightarrows_A \mathcal{M}^H$, given by $M \mapsto A \otimes_R M \cong AM$ and $N \mapsto N^{\mathrm{co}H}$, for $M \in {}_R\mathcal{M}$ and $N \in {}_A\mathcal{M}^H$,

where $\mathcal{M}_A^H$ and ${}_A\mathcal{M}^H$ are the categories of (A, H)-Hopf modules taken on the appropriate sides.

Definition 1.1: (1) An ideal I of R is called H-**stable** if $IA = AI$.

(2) For any ideal I of R, $(I : H)$ denotes the largest H-stable ideal of R contained in I.

Note that $(I : H)$ exists since a sum of H-stable ideals is again H-stable.

Remark 1.2: (i) The definition of H-stable above is a natural one. For when $A = R\#_\sigma H$, it is shown in [MS, 1.3] that if an ideal I of R is H-stable in the usual sense (that is , $H \cdot I \subset I$) and the antipode of H is bijective, then I is H-stable in the sense of Definition 1.1. We note that it is false in general that $IH = HI$ for any H-stable ideal I [MS, 2.6]. (ii) When σ is trivial, that is A is a smash product, then I is stable in the usual sense provided $AI \subset IA$. In fact this seemingly weaker property is equivalent to $AI = IA$, even in the general Galois situation: for, assume that $AI \subset IA$ and let $M = IA$. Then $M \in {}_A\mathcal{M}^H$, and so the category equivalence (ii) implies $(IA) \cap R = I$ and $IA = A(IA \cap R) = AI$.

The category equivalences have immediate consequences for ideals of A and R. For any subspace J of A, let $\widetilde{J}$ denote the largest H-subcomodule of A contained in J.

LEMMA 1.3: *Consider $A \in_A \mathcal{M}^H$ and $A \in \mathcal{M}_A^H$.*

(1) *If J is any ideal of A, then $\widetilde{J} \cap R = J \cap R$ is an H-stable ideal of R. Moreover, $(J \cap R)A = A(J \cap R) = \widetilde{J}$.*

(2) *If I is any ideal of R, then $(IA) \cap R = (AI) \cap R = I$.*

(3) *If I and I' are ideals of R, then $(I \cap I')A = IA \cap I'A$ and $A(I \cap I') = AI \cap AI'$. Consequently $(I : H) \cap (I' : H) = ((I \cap I') : H)$.*

(4) *There is a bijection of sets*

$$\{H\text{-stable ideals of } R\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{\text{ideals of } A \text{ which are } H\text{-subcomodules}\}$$

given by Φ: $I \mapsto IA = AI$, I an H-stable ideal of R, and Ψ: $J \mapsto J \cap R$, J an ideal of A and an H-subcomodule. These bijections preserve sums, intersections, and products.

Proof: (1) Clearly $\widetilde{J} \cap R \subset J \cap R$. Conversely, consider $J_1 = A(J \cap R)A$; it is an ideal of A contained in J which is an H-subcomodule, and so $J_1 \subset \widetilde{J}$. Thus $J \cap R \subset \widetilde{J}$, and so $J \cap R = \widetilde{J} \cap R$. Now $\widetilde{J} \in \mathcal{M}_A^H$, and so $\widetilde{J} = (\widetilde{J} \cap R)A = (J \cap R)A$ by the category equivalence (i); similarly $\widetilde{J} = A(J \cap R)$ using (ii).

(2) Since $IA \in \mathcal{M}_A^H$ and $AI \in {}_A\mathcal{M}^H$, (2) is clear from (i) and (ii).

(3) The first statement follows from the fact that (i) and (ii) preserve intersections, and the second follows from the first.

(4) Given $I \lhd R$ which is H-stable, let $J = AI = IA$; clearly $\widetilde{J} = J$, and $J \cap R = I$ by (2); thus $\Psi \circ \Phi = \mathrm{id}$. Given $J = \widetilde{J} \lhd A$, let $I = J \cap R$; then I is H-stable and $J = IA$ by (1). Thus $\Phi \circ \Psi = \mathrm{id}$. Φ preserves intersections by (3), and Ψ preserves them, since Φ is surjective. Similarly for sums and products: if I and I' are H-stable, then

$$(I + I')A = IA + I'A = AI + AI' = A(I + I'),$$

and

$$(II')A = IAI'A = IAI' = AII'.$$

Thus $I + I'$ and II' are H-stable, and Φ preserves sums and products. Since Φ is a bijection, it follows that Ψ also preserves sums and products. ∎

Now assume in addition that H is finite-dimensional . Then the coaction of H on A dualizes to give an action of H^* on A, in which H-subcomodules correspond to H^*-submodules. In particular $\widetilde{J} = (J : H^*)$, and Lemma 1.3(4) gives a correspondence between H-stable ideals of R and H^*-stable ideals of A. This correspondence can be strengthened by using the smash product $A\#H^*$.

Since $R \subset A$ is H-Galois, it follows by [KT] that $A\#H^* \cong \mathrm{End}(A_R)$. Thus the bimodule $M = {}_{A\#H^*}A_R$ gives a Morita equivalence. The next theorem extends [MS, Theorem 7.2], where the analogous result was shown when A was a crossed product.

THEOREM 1.4: *The Morita equivalence between R and $A\#H^*$ via M as above defines a bijection ϕ from ideals of R to ideals of $A\#H^*$, preserving containments, intersections, and products, such that if*

$$\phi: I \mapsto I'$$

for I an ideal of R and I' an ideal of $A\#H^*$, then

$$(I : H) \mapsto (I' \cap A)\#H^* = (I : H)A\#H^*.$$

Consequently $(I : H)A = I' \cap A$ and $(I : H) = I' \cap R$.

Proof: (a) The Morita equivalence gives a bijection between ideals I of R and $(A\#H^*, R)$-subbimodules of A which maps I to AI. Similarly, there is a bijection between ideals J of $A\#H^*$ and $(A\#H^*, R)$-subbimodules of A which maps J to $J \cdot A$. Hence there is a bijection ϕ between ideals of R and of $A\#H^*$ in which I corresponds to I' if and only if $AI = I' \cdot A$.

(b) Next we show that under ϕ, H-stable ideals of R correspond to ideals of $A\#H^*$ which are H^*-subcomodules. Let I be an H-stable ideal of R. Then

$$(IA\#H^*) \cdot A = (IA)A = IA = AI$$

since I is H-stable. Thus $I' = \phi(I) = IA\#H^*$, clearly an H^*-subcomodule. Conversely, assume $I' \lhd A\#H^*$ and I' is an H^*-subcomodule. Applying the correspondence (ii) to the H^*-Galois extension $A \subset A\#H^*$, we see $I' = (I' \cap A)\#H^*$. Now $I' \cap A$ is an H^*-stable ideal of A, so an H-subcomodule, and thus by Lemma 1.3(1),

$$I' \cap A = ((I' \cap A) \cap R)A = (I' \cap R)A.$$

Hence

$$I' \cdot A = ((I' \cap R)A\#H^*) \cdot A = (I' \cap R)A = A(I' \cap R),$$

where the last equality holds since $I' \cap R$ is H-stable, by Lemma 1.3(1). Therefore by definition of the bijection ϕ in (a), $I = \phi^{-1}(I') = I' \cap R$ is H-stable .

(c) By (b), the largest H-stable ideal $(I : H)$ in an ideal I of R corresponds to the largest ideal in I' which is an H^*-subcomodule, that is, to $(I' \cap A)\#H^*$. Finally, applying (b) to $(I : H)$, we see that ϕ maps $(I : H)$ to $(I : H)A\#H^*$. Thus $I' \cap A = (I : H)A$, and also $I' \cap R = ((I : H)A) \cap R = (I : H)$ by Lemma 1.3(2). ∎

The next corollary will be useful in constructing prime and semiprime ideals of R.

Corollary 1.5: *Let L be an H-stable ideal of R and choose $x \in L$. Then there exists a finitely generated H-stable ideal I of R such that $x \in I \subset L$.*

Proof: We first claim that for any ideals $\mathcal{J} \subset \mathcal{L}$ of $A\#H^*$, such that $\mathcal{J}$ is finitely generated and $\mathcal{L}$ is an H^*-subcomodule, there exists an ideal $\widetilde{\mathcal{J}}$ of $A\#H^*$, with $\mathcal{J} \subset \widetilde{\mathcal{J}} \subset \mathcal{L}$, such that $\widetilde{\mathcal{J}}$ is both finitely generated and an H^*-subcomodule.

Assume $\mathcal{J} = \sum_{i=1}^{n} Sx_iS$, where we let $S = A\#H^*$. Then there exists a finite-dimensional H^*-subcomodule V of $\mathcal{L}$ such that all $x_i \in V$ (by the local finiteness of comodules). Setting $\widetilde{\mathcal{J}} := SVS$ proves the claim.

We now apply the Morita correspondence in Theorem 1.4. Let $J = RxR$; then $J \subset L$ is a finitely generated ideal of R, and consequently $\mathcal{J} = \phi(RxR)$ is a finitely generated ideal of S. Moreover $\mathcal{J} \subset \mathcal{L} := \phi(L)$, and by Theorem 1.4 we have $\mathcal{L}$ is an H^*-subcomodule since L is H-stable . Thus there exists $\widetilde{\mathcal{J}}$ as above, $\mathcal{J} \subset \widetilde{\mathcal{J}} \subset \mathcal{L}$, with $\widetilde{\mathcal{J}}$ a finitely generated ideal and H^*-subcomodule. Thus by Theorem 1.4, $I := \phi^{-1}(\widetilde{\mathcal{J}})$ is finitely generated and H-stable ; moreover $x \in RxR = J \subset I$, and $I \subset L = \phi^{-1}(\mathcal{L})$. ∎

We close this section with a consideration of what happens to our set-up modulo H-stable ideals. Thus choose an H-stable ideal I of R and set $\bar{R} := R/I$. Since $IA = AI$ is an ideal of A, we may define $\bar{A} := A/IA$; since $IA \cap R = I$ by Lemma 1.3(2), it follows that $\bar{R}$ can be canonically embedded into $\bar{A}$.

Lemma 1.6: *Let I, $\bar{R}$, and $\bar{A}$ be as above. Then under the induced H-comodule structure on $\bar{A}$, $\bar{R} \subset \bar{A}$ is a faithfully flat right H-Galois extension.*

Proof: Note that $\bar{A}$ is indeed an H-comodule since IA is an H-subcomodule of A; it is clear that $\bar{R} \subset \bar{A}^{\mathrm{co}H}$.

Moreover $\bar{R} \otimes_R A = R/I \otimes_R A \cong A/IA \cong \bar{A}$. Now for any right $\bar{R}$-module M, $M \otimes_{\bar{R}} \bar{A} = M \otimes_{\bar{R}} (\bar{R} \otimes_R A) \cong M \otimes_R A$. Thus $_{\bar{R}}\bar{A}$ is faithfully flat since $_RA$ is faithfully flat.

Since $R \subset A$ is H-Galois, the Galois map $A \otimes_R A \to A \otimes H$, given by $x \otimes y \mapsto \sum xy_0 \otimes y_1$, is a bijection. Thus

$$\bar{R} \otimes_R A \otimes_R A \xrightarrow{\cong} \bar{R} \otimes_R A \otimes H.$$

By the remark above, this gives

$$\bar{A} \otimes_{\bar{R}} \bar{A} \xrightarrow{\cong} \bar{A} \otimes_{\bar{R}} \bar{R} \otimes_R A \xrightarrow{\cong} \bar{A} \otimes_R A \xrightarrow{\cong} \bar{A} \otimes H,$$

via the induced map $\bar{x} \otimes \bar{y} \mapsto \sum \overline{xy_0} \otimes y_1$. Now consider the diagram

$$\begin{array}{ccccc} \bar{R} & \subset & \bar{A} & \overset{i_1}{\underset{i_2}{\rightrightarrows}} & \bar{A} \otimes_{\bar{R}} \bar{A} \\ \cap & & \| & & \downarrow\cong \\ \bar{A}^{\mathrm{co}H} & \subset & \bar{A} & \rightrightarrows & \bar{A} \otimes H \end{array}$$

The top row is exact since $_{\bar{R}}\bar{A}$ is faithfully flat , and the bottom row is exact by the definition of $\bar{A}^{\mathrm{co}H}$. Since the diagram commutes and we have shown that the vertical arrow is an isomorphism, we must have $\bar{R} = \bar{A}^{\mathrm{co}H}$. Thus $\bar{R} \subset \bar{A}$ is H-Galois. ∎

Remark 1.7: (1) If $Q \lhd R$ is H-stable , then its image $\bar{Q}$ is H-stable in $\bar{R}$, since $\bar{Q}\bar{A} = \overline{QA} = \overline{AQ} = \bar{A}\bar{Q}$. Conversely, if $J \lhd \bar{R}$ is H-stable and Q is the inverse image of J in R, then also Q is H-stable . For, $\bar{Q}\bar{A} = \bar{A}\bar{Q}$ implies $QA \subset AQ + AI = AQ$; similarly $AQ \subset QA$.

(2) In the special case when $A = R\#H$ and R is an H-module algebra, Lemma 1.6 has an easy proof. For then $IA = I\#H$, and thus $\bar{A} = R\#H/I\#H \cong R/I\#H = \bar{R}\#H$, and clearly $\bar{R} \subset \bar{R}\#H$ is H-Galois.

2. H-prime ideals and equivalence relations on Spec

In this section we extend the usual notion of H-prime ideals to our set-up of a faithfully flat H-Galois extension $R \subset A$ and consider correspondences between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(A)$. We also assume that H is finite-dimensional, so that there is an H^*-action on A.

Definition 2.1: An H-stable **ideal** I of R is H**-prime** if $I \neq R$ and whenever $LM \subset I$, for H-stable ideals L, M of R, then $L \subset I$ or $M \subset I$. R itself is an H-prime ring if $R \neq \{0\}$ and if $\{0\}$ is an H-prime ideal of R. The set of all H-prime ideals of R will be denoted by H-Spec(R).

To avoid confusion, we will usually write $Q \in \mathrm{Spec}(R)$ and $P \in \mathrm{Spec}(A)$.

LEMMA 2.2:

(1) *The bijections in Lemma 1.3(4) restrict to give bijections*

$$H\text{-Spec}(R) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} H^*\text{-Spec}(A),$$

where as before $\Phi\colon I \mapsto IA$ *and* $\Psi\colon J \mapsto J \cap R$.

(2) *The map* $f\colon \mathrm{Spec}(R) \to H\text{-Spec}(R)$ *given by* $Q \mapsto (Q : H)$ *is well-defined and surjective.*

(3) *The map* $g\colon \mathrm{Spec}(A) \to H\text{-Spec}(R)$ *given by* $P \mapsto P \cap R$ *is well-defined and surjective.*

Proof: (1) We only need to show that $\mathrm{Im}(\Phi)$ and $\mathrm{Im}(\Psi)$ are in the correct subsets. If $I \in H\text{-Spec}(R)$, let $J = \Phi(I)$; J is H^*-stable by Lemma 1.3(4). If U, V are H^*-stable ideals of A with $UV \subset J$, then $\Psi(U)\Psi(V) \subset \Psi(J) = I$ since Ψ preserves products. Since I is H-prime , and $\Psi(U)$, $\Psi(V)$ are H-stable by 1.3(4), either $\Psi(U) \subset I$ or $\Psi(V) \subset I$. But then $U \subset J$ or $V \subset J$, and so J is H^*-prime.

The converse is similar, using that Φ preserves products and sends H-stable ideals to H^*-stable ideals.

(2) It is easy to see that $(Q : H)$ is H-prime , for if $LM \subset (Q : H)$, for H-stable ideals L, M of R, then $LM \subset Q$ and so $L \subset Q$ or $M \subset Q$. But then $L \subset (Q : H)$ or $M \subset (Q : H)$. To see that f is surjective requires some work. Thus, let $I \in H\text{-Spec}(R)$.

We consider the set $\mathcal{I} = \{J \lhd R \mid (J : H) = I\}$. $\mathcal{I} \neq \emptyset$ since $I \in \mathcal{I}$. We claim that $\mathcal{I}$ is closed under ascending chains. Let J_i, i in some index set, be an ascending chain in $\mathcal{I}$ and let J be their union. Since $(J_i : H) = I$ for all i, clearly $I \subset (J : H)$. Conversely, choose $x \in (J : H)$. By Corollary 1.5 with $L = (J : H)$, there exists a finitely generated H-stable ideal M of R with $x \in M \subset (J : H)$. Since $M \subset J$ and M is finitely generated , $M \subset J_i$ for some i. But then $M \subset (J_i : H) = I$, and so $x \in I$. Thus $(J : H) = I$ and $J \in \mathcal{I}$. We may now apply Zorn's lemma and choose a maximal element $Q \in \mathcal{I}$.

We claim that Q is a prime ideal of R. For if L, M are ideals of R with $LM \subset Q$, we may assume $L, M \supsetneqq Q$. By the maximality of Q, $(L : H) \supsetneqq I$ and

$(M : H) \supsetneq I$. However $(L : H)(M : H) \subset (Q : H) = I$, since the product of H-stable ideals is H-stable ; this contradicts $I \in H\text{-Spec}(R)$. Thus Q is prime.

(3) We give a direct argument to see that g is well-defined. For if $P \in \text{Spec}(A)$ and $LM \subset P \cap R$, where L, M are H-stable ideals of R, then $(LA)(MA) = LMA \subset P$; since P is prime and $LA, MA \lhd A$, either $LA \subset P$ or $MA \subset P$. But then $L \subset P \cap R$ or $M \subset P \cap R$, so $P \cap R$ is H-prime .

To see that g is surjective, choose $I \in H\text{-Spec}(R)$, and let

$$\mathcal{I} = \{J \lhd A \mid J \cap R = I\}.$$

$\mathcal{I} \neq \emptyset$ since $IA \in \mathcal{I}$, and $\mathcal{I}$ is closed under ascending unions, so we may choose P maximal in $\mathcal{I}$. We claim that P is prime. If $LM \subset P$, for $L, M \lhd A$ with $L, M \supsetneq P$, then $(L \cap R)(M \cap R) \subset P \cap R = I$. Since $L \cap R$ and $M \cap R$ are H-stable , either $L \cap R \subset I$ or $M \cap R \subset I$. But then $L, M \supsetneq P$ implies $L \cap R = I$ or $M \cap R = I$, a contradiction to the maximality of P (note this argument did not require H to be finite-dimensional). ∎

We next set up a bijection between certain equivalence classes of primes of R and of A.

Definition 2.3: (1) $P \in \text{Spec}(A)$ **lies over** $Q \in \text{Spec}(R)$ if $P \cap R = (Q : H)$.

(2) For $Q_1, Q_2 \in \text{Spec}(R)$, define $Q_1 \sim_H Q_2 \iff (Q_1 : H) = (Q_2 : H)$.

(3) For $P_1, P_2 \in \text{Spec}(A)$, define $P_1 \sim_{H^*} P_2 \iff (P_1 : H^*) = (P_2 : H^*)$.

We remark that for an arbitrary ring extension $R \subset A$, to say that P lies over Q would usually mean that Q is minimal over $P \cap R$. We will discuss in Corollary 4.7 when our condition is equivalent to this one.

COROLLARY 2.4: *Let $R \subset A$ be faithfully flat H-Galois, where H is finite-dimensional. Then $\sim_H$ and $\sim_{H^*}$ are equivalence relations, and there is a bijection*

$$\text{Spec}(R)/\sim_H \xrightarrow{\widetilde{\Phi}} \text{Spec}(A)/\sim_{H^*}$$

where for $Q \in \text{Spec}(R)$ and $P \in \text{Spec}(A)$, $\widetilde{\Phi}$: $[Q] \mapsto [P]$ if and only if P lies over Q.

Proof: Clearly $\sim_H$ and $\sim_{H^*}$ are equivalence relations. Moreover, Lemma 2.2(2) induces bijections

$$\widetilde{f}: \text{Spec}(R)/\sim_H \to H\text{-Spec}(R), \text{ via } [Q] \mapsto (Q : H)$$

and

$$\widetilde{g}: \text{Spec}(A)/\sim_{H^*} \mapsto H^*\text{-Spec}(A), \text{ via } [P] \mapsto (P : H^*).$$

By 2.2(1), Φ: H-Spec$(R) \to H^*$-Spec(A) is also a bijection. Thus $\widetilde{\Phi} := \widetilde{g}^{-1} \circ \Phi \circ \widetilde{f}$ is a bijection between the desired quotient sets. Now

$$\begin{aligned}\widetilde{\Phi}([Q]) = [P] &\iff \widetilde{f}([Q]) = \Phi^{-1} \circ \widetilde{g}[P] = \Psi \circ \widetilde{g}[P] \\ &\iff (Q : H) = \Psi((P : H^*)) = P \cap R.\end{aligned}$$ ∎

We wish to refine this correspondence on Spec. As we will see in Section 3, it is only necessary to consider stability of primes under the coradical of H. As a first step, we recall a result of Chin for pointed Hopf algebras [Ch90]; part (1) of the lemma is [Ch90, Lemma 2.2(i)] and part (2) is implicit in [Ch90, 2.2(ii)]. We give the proof, as it is very short.

Lemma 2.5 (Chin): *Let H be a finite-dimensional pointed Hopf algebra and A an H-module algebra. Let $G = G(H)$ denote the set of group-like elements in H.*

(1) *For any ideal P of A,*

$$\left(\bigcap_{x \in G} (x \cdot P) \right)^m \subset (P : H)$$

for some $m \leq \dim H$.

(2) *For any $P_1, P_2 \in$ Spec(A), $(P_1 : H) = (P_2 : H) \iff P_2 = x \cdot P_1$, for some $x \in G$. Thus*

$$\mathrm{Spec}(A)/\sim_H = \mathrm{Spec}(A)/G,$$

where Spec$(A)/G$ is the set of G-orbits in Spec(A).

Proof: (1) Note that $J = \bigcap_{x \in G} x \cdot P$ is the largest G-stable ideal of A in P. Let $H_0 \subset H_1 \subset \cdots \subset H_m = H$ denote the coradical filtration of H; $H_0 = kG$ since H is pointed. We claim that for all $j \geq 0$, $H_j \cdot J^n \subset J$ for any $n > j$. This is clear for $j = 0$. Assuming it is true for all $i < j$, and $n > j$, then

$$H_j \cdot (J^n) = H_j \cdot (J \cdot J^{n-1}) \subset \sum_{i=0}^{j} (H_i \cdot J)(H_{j-i} \cdot J^{n-1}) \subset J.$$

Thus $H \cdot J^{m+1} = H_m \cdot J^{m+1} \subset J$. It follows that $J^{m+1} \subset (J : H) \subset (P : H)$.

(2) Clearly $P_2 = x \cdot P_1$ implies $(P_1 : H) = (P_2 : H)$. So assume that $(P_1 : H) = (P_2 : H)$. By (1),

$$(\bigcap_{x \in G} x \cdot P_1)^m \subset (P_1 : H) = (P_2 : H) \subset P_2.$$

It follows that $y \cdot P_1 \subset P_2$ for some $y \in G$. Similarly $z \cdot P_2 \subset P_1$ for some $z \in G$. Then $yz \cdot P_2 \subset y \cdot P_1 \subset P_2$; since G is finite, $yz \cdot P_2 = y \cdot P_1 = P_2$. This proves (2). ∎

COROLLARY 2.6: *Let $R \subset A$ be a faithfully flat H-Galois extension, with H finite-dimensional , such that H^* is pointed and A is a centralizing extension of R. If $G = G(H^*)$, then there is a bijection*

$$\mathrm{Spec}(A)/G \xrightarrow{\widetilde{\Psi}} \mathrm{Spec}(R), \quad [P] \mapsto P \cap R.$$

Proof: First, since A is a centralizing extension of R, all ideals of R are H-stable, and thus H-Spec(R) = Spec(R). Equivalently, Spec$(R)/\sim_H$= Spec(R). By Lemma 2.5, Spec$(A)/\sim_{H^*}$= Spec$(A)/G$. The corollary now follows from Corollary 2.4. ∎

The previous corollary has interesting applications to Hopf algebras A with a central sub Hopf algebra R of finite index. If A is faithfully flat over R, then $R \subset A$ is faithfully flat H-Galois, $H := A/AR^+$, with coaction $A \to A \otimes H$, $a \mapsto \sum a_1 \otimes \bar{a}_2$, by [T, Th. 1] and [S4, 1.6]. A is always faithfully flat over R in case R is noetherian by [S4, Th. 3.3]. The next lemma describes another class of examples using duality.

LEMMA 2.7: *Let U be a pointed Hopf algebra , $I \subset U$ a Hopf ideal and $K := U^{\mathrm{co}U/I}$. Assume K is finite-dimensional and I is cocentral in U, that is for all $x \in U$,*

$$\sum (x_1 \otimes x_2 - x_2 \otimes x_1) \in I \otimes I.$$

Let R be the image of $(U/I)^0$ in U^0 under the Hopf algebra map dual to the canonical map $U \to U/I$. Let $A \subset U^0$ be any sub Hopf algebra containing R.

Then $R \subset A$ is a central sub Hopf algebra , the quotient Hopf algebra $H := A/AR^+$ is finite-dimensional and H^* is pointed, A is a finitely generated projective R-module and R is an R-direct summand in A. In particular, $R \subset A$ is faithfully flat H-Galois.

Proof: (1) To see that R is central in U^0, let $f \in U^0$, $g \in R$ and $x \in U$, and let $U \to U/I$, $x \mapsto \bar{x}$, be the quotient map. Then

$$(gf)(x) = \sum g(\bar{x}_1)f(x_2) = \sum f(x_2)g(\bar{x}_1) = \sum f(x_1)g(\bar{x}_2) = (fg)(x),$$

since I is cocentral, hence $\sum x_1 \otimes \bar{x}_2 = \sum x_2 \otimes \bar{x}_1$.

Let ϕ: $U^0 \to K^0$ be the restriction map. We want to show that the kernel of ϕ is a conormal Hopf ideal of U^0. Since I is cocentral hence conormal in U, K is a normal sub Hopf algebra of U [S4, 1.3]. Let $f \in U^0$ with $\phi(f) = f\,|_K = 0$. Then for all $x \in U$ and $y \in K$, $\sum x_1 y S(x_2) \in K$ by normality, hence

$$\sum f_1(x_1)(Sf_3)(x_2)f_2(y) = f(\sum x_1 y S(x_2)) = 0.$$

Thus $\sum f_1(Sf_3)\otimes f_2 \in U^0\otimes \mathrm{Ker}(\phi)$, and similarly $\sum f_2\otimes(Sf_1)f_3 \in \mathrm{Ker}(\phi)\otimes U^0$. Hence $\mathrm{Ker}(\phi)$ is conormal.

We finally show that R is the Hopf kernel of ϕ, that is $R = (U^0)^{\mathrm{co}K^0}$. By definition, $(U^0)^{\mathrm{co}K^0}$ is the set of all $f \in U^0$ with $\sum f_1 \otimes \phi(f_2) = f\otimes 1$, i.e. $\sum f_1(x)f_2(y) = f(x)\varepsilon(y)$ or $f(xy) = f(x\varepsilon(y))$ for all $x\in U$ and $y \in K$; equivalently $f(UK^+) = \{0\}$. But $I = UK^+$ since U is pointed [Ma91, Th. 1.3] and hence $R = (U^0)^{\mathrm{co}K^0}$ since $f \in R$ if and only if $f(I) = \{0\}$.

(2) By Part (1) of the proof, R is a central sub Hopf algebra of $A \subset U^0$, the kernel J of the restriction map $A \to K^0$ is conormal and $R = A^{\mathrm{co}A/J}$. By assumption, K is finite-dimensional and pointed. Then $H := A/J$ is a finite-dimensional sub Hopf algebra of $K^0 = K^*$ and its dual is a quotient of K, hence pointed. Note that the quotient map induces the natural Galois map $A\otimes_R A \to A\otimes A/J$, which is surjective. Since A/J is finite-dimensional and R is commutative, it follows from [KT] that $R \subset A$ is A/J-Galois, A is finitely generated projective as an R-module and R is an R-direct summand in A. Moreover, A is left and right faithfully coflat over A/J by [S4, 2.1(1)], hence $J = AR^+$ by [T, Theorem 2]. ∎

As a special case of the preceding lemma we now consider the Frobenius–Lusztig kernels. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra, l an odd integer (prime to 3 in case $\mathfrak{g}$ has a G_2 component) and ε a primitive l-th root of 1 in $\mathbb{C}$. Let $U_\varepsilon(\mathfrak{g})$ be Lusztig's quantum enveloping algebra over $\mathbb{Q}(\varepsilon)$ defined by extending scalars via $\mathbb{Z}[v,v^{-1}] \to \mathbb{Q}(\varepsilon)$, $v\mapsto\varepsilon$, v an indeterminate, from Lusztig's form [Lu, 1.3]. We want to apply 2.7 to the Frobenius homomorphism $Fr : U_\varepsilon(\mathfrak{g}) \to U(\mathfrak{g})$ of [Lu, 8.10]. Here, $U(\mathfrak{g})$ is the usual enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(\varepsilon)$. Fr is a surjective Hopf algebra homomorphism of pointed Hopf algebras. Consider the dual Hopf algebra map $Fr^0 : U(\mathfrak{g})^0 \to U_\varepsilon(\mathfrak{g})^0$. Let $F_0[G]$ be the image of Fr^0. Then $F_0[G] \cong U(\mathfrak{g})^0$ is the usual coordinate algebra of the connected, simply connected, semisimple algebraic group G with Lie algebra $\mathfrak{g}$. The quantum coordinate algebra $\mathcal{O}_\varepsilon[G]$ is a sub Hopf algebra of $U_\varepsilon(\mathfrak{g})^0$ containing $F_0[G]$ defined by a certain class of finite-dimensional representations [Lu, 8.17], [A, 3.4.5], [DL, 6.4,4.1].

COROLLARY 2.8: *$F_0[G] \subset \mathcal{O}_\varepsilon[G]$ is a central sub Hopf algebra , $\mathcal{O}_\varepsilon[G]$ is finitely generated and projective over $F_0[G]$, $F_0[G]$ is an $F_0[G]$-direct summand in $\mathcal{O}_\varepsilon[G]$, and $H := \mathcal{O}_\varepsilon[G]/\mathcal{O}_\varepsilon[G]F_0[G]^+$ is finite-dimensional such that H^* is pointed. In particular, $F_0[G] \subset \mathcal{O}_\varepsilon[G]$ is faithfully flat H-Galois, and*

$$\mathrm{Spec}(\mathcal{O}_\varepsilon[G])/\chi \xrightarrow{\cong} \mathrm{Spec}(F_0[G]), \quad [P]\mapsto P\cap F_0[G],$$

where $\chi := \mathrm{Alg}(H, \mathbb{Q}(\varepsilon))$ *is the character group of* H.

Proof: Let I be the kernel of the Frobenius homomorphism and $U := U_\varepsilon(\mathfrak{g})$. It is easy to see that I is a cocentral Hopf ideal in U by checking the generators of U as in [DL, 6.4]. The Frobenius–Lusztig kernel u is the finite-dimensional sub Hopf algebra of U defined in [Lu, 8.2]. Then $I = u^+Uu^+$ by [Lu, 8.16]. It is shown in [A, 3.4.2] that u is a normal sub Hopf algebra of U. Hence $u = U^{\mathrm{co}\, U/I}$ [A, 3.4.1]. Now all the assumptions of 2.7 are verified. Thus the corollary follows from 2.7 and 2.6 with $R := F_0[G]$ and $A := \mathcal{O}_\varepsilon[G]$. ∎

In [DL, 7.2], projectivity of the $F_0[G]$-module $\mathcal{O}_\varepsilon[G]$ was shown in a way completely different from the above general Hopf algebra arguments. Concerning the prime ideal correspondence in 2.8, a very similar result for $G = SL(n)$ was shown in [L1] by direct calculations using generators and relations for the quantum coordinate algebra of $SL(n)$. In [DP, 4.10], the correspondence is shown for l prime to the bad primes of the root system; the proof involves an analysis of certain Azumaya algebras. Finally in [L2] a different proof of the corespondence for general G is given using results from Noetherian ring theory.

3. Reducing to the coradical

In this section we prove a greatly generalized version of Chin's result, Lemma 2.5, in a different way. We show that H-Spec(R) can be identified with H_0-Spec(R), where $R \subset A$ is a faithfully flat H-Galois extension and H_0 is the coradical of H; in general H_0 is not a sub Hopf algebra. To this end we first define C-stable ideals and C-prime ideals for subcoalgebras C of H.

As above let $R \subset A$ be a faithfully flat H-Galois extension and let H be finite-dimensional .

Definition 3.1: Let $C \subset H$ be a subcoalgebra. Define $A(C) := \rho^{-1}(A \otimes C)$, an R-subbimodule of A. An ideal I in R is called C**-stable** if $IA(C) = A(C)I$. Let $(I : C)$ denote the largest C-stable ideal in R which is contained in I. A C-stable ideal I in R, $I \neq R$, is called C**-prime** , if whenever $KL \subset I$ for K, L C-stable ideals of R, then $K \subset I$ or $L \subset I$. C-Spec(R) is the set of all C-prime ideals in R.

Remark 3.2: (1) Let R be an H-module algebra, $A = R\#H$ the smash product and $C \subset H$ a subcoalgebra. Then $A(C) = R\#C$. Any C-stable ideal I in R is stable under the action of C, i.e. $c \cdot r \in I$ for all $c \in C$ and $r \in I$. Conversely, if $S(C) = C$, any ideal in R which is stable under the action of C is also C-stable. This follows from the identity $r\#c = \sum (1\#c_2)(S^{-1}(c_1) \cdot r)$ in $R\#C$.

(2) Let H_0 be the coradical of H and I an ideal in R. Then I is H_0-stable iff I is C-stable for all simple subcoalgebras C of H.

Proof: Let $(C_i)_{i\in J}$ be the set of all simple subcoalgebras of H. Then $H_0 = \bigoplus_{i\in J} C_i$. Hence $A(C) = \bigoplus_i A(C_i)$ is a decomposition into R-bimodules. Thus $IA(C) = A(C)I$ iff $IA(C_i) = A(C_i)I$ for all i. ∎

In the sequel we will need the cotensor product $V\square_C W$ of a right C-comodule V and a left C-comodule W over any coalgebra C. Recall that $V\square_C W$ is the equalizer of the maps $\Delta_V \otimes \mathrm{id}$ and $\mathrm{id}\otimes\Delta_W$ from $V\otimes W$ to $V\otimes C\otimes W$. Note that Δ induces an isomorphism $C \xrightarrow{\cong} H\square_H C$. Similarly, in 3.1, ρ induces an isomorphism $A(C) \xrightarrow{\cong} A\square_H C$.

LEMMA 3.3: *Let $C\subset H$ be a sub coalgebra and I an ideal in R.*

(1) *If I is H-stable then I is C-stable .*

(2) $((I:C):H) = (I:H)$.

Proof: (1) The Galois map $A\otimes_R A \xrightarrow{\cong} A\otimes H$, $x\otimes y\mapsto \sum xy_0\otimes y_1$, is right H-colinear where the H-comodule structures are $\mathrm{id}\otimes\rho$ and $\mathrm{id}\otimes\Delta$. By cotensoring with $\square_H C$ we get an isomorphism of R-bimodules $A\otimes_R A(C) \xrightarrow{\cong} A\otimes C$. Here, $A\otimes_R A(C)$ is an R-bimodule by left multiplication on A and right multiplication on $A(C)$, and $A\otimes C$ is an R-bimodule via the bimodule structure of A. Note that we used the bijectivity of the natural map

$$A\otimes_R (A\square_H C) \xrightarrow{\cong} (A\otimes_R A)\square_H C$$

which holds since A is right R-flat.

Since I is H-stable we have $AI = IA$ and hence $(A\otimes_R A(C))I = I(A\otimes_R A(C))$. We want to conclude that $A(C)I = IA(C)$. For any R-submodule M of A let $\Im(A\otimes_R M)$ denote the image of $A\otimes_R M$ in $A\otimes_R A$ defined by the inclusion map. This notation is also used for submodules of $A\otimes_R M$. Then $\Im(I(A\otimes_R A(C))) = \Im(A\otimes_R IA(C))$ and $\Im((A\otimes_R A(C))I) = \Im(A\otimes_R A(C)I)$.

From $(A\otimes_R A(C))I = I(A\otimes_R A(C))$ we get $\Im(A\otimes_R X) = \Im(A\otimes_R Y)$, where $X := IA(C)$ and $Y := A(C)I$. Since A is right faithfully flat over R we conclude that $X = Y$, and I is C-stable .

To see that $\Im(A\otimes_R X) = \Im(A\otimes_R Y)$ implies $X = Y$ for all R-submodules X, Y of A we can assume that $X\subset Y$ (replace Y by $X+Y$). Then $A\otimes_R X\to A\otimes_R Y$ is surjective, hence $X = Y$ by faithful flatness.

(2) Since $(I:C)$ is an ideal in I, $((I:C):H)$ is contained in $(I:H)$. To prove the other containment let $L\subset I$ be an H-stable ideal of R. By (1), L

is C-stable . Hence $L \subset (I : C)$ and $L = (L : H) \subset ((I : C) : H)$. Thus $(I : H) \subset ((I : C) : H)$. ∎

COROLLARY 3.4: *Let $C \subset H$ be a sub coalgebra. Then*

$$f\colon C\text{-Spec}(R) \to H\text{-Spec}(R), \quad f(I) := (I : H),$$

is well-defined and a surjection.

Proof: To see that f is well-defined let $I \in C$-Spec(R) and let K, L be H-stable ideals in R such that $KL \subset (I : H)$. Then $KL \subset I$, and K, L are C-stable by 3.3(1). Hence $K \subset I$ or $L \subset I$. Since K and L are H-stable , this implies $K \subset (I : H)$ or $L \subset (I : H)$.

To prove the surjectivity of f, let $I \in H$-Spec(R). By 2.2(2) there is a prime $P \in$ Spec(R) such that $(P : H) = I$. By 3.3(2), $(P : H) = ((P : C) : H)$, and $(P : C) \in C$-Spec(R) by the proof of 2.2(2). Thus $I = f((P : C))$. ∎

Now let $H_0 \subset H_1 \subset \cdots$ be the coradical filtration of H (cf. [M, 5.2]), and define $A_n := A(H_n)$ for $n \geq 0$. Then $A_0 \subset A_1 \subset \cdots$ is a filtration of R-subbimodules in A. Note that $H_n \subset H_{n+1}$ are subcoalgebras of H, in particular left (and right) sub H-comodules. Then the quotient H-comodule H_{n+1}/H_n is in fact an H_0-comodule since $\Delta(H_{n+1}) \subset \sum_{i+j=n+1} H_i \otimes H_j$ (cf. [M, 5.2.2]).

The next lemma contains the crucial observation (cf. [S2, 1.4, 2.1] for the case when H is pointed).

LEMMA 3.5: *For all $n \geq 0$, the map*

$$\phi\colon A_{n+1}/A_n \to A_0 \Box_{H_0} (H_{n+1}/H_n), \quad \phi(\bar{x}) := \sum x_0 \otimes \bar{x}_1,$$

is an isomorphism of R-bimodules. Here, the cotensor product is an R-bimodule via the bimodule structure of A_0.

Proof: Let $\bar{A}_{n+1} := A_{n+1}/A_n$ and $\bar{H}_{n+1} := H_{n+1}/H_n$. Consider the following diagram

$$\begin{array}{ccccc} \bar{A}_{n+1} & \xrightarrow{i} & A \otimes_R \bar{A}_{n+1} & \overset{i_1\otimes \mathrm{id}}{\underset{i_2\otimes \mathrm{id}}{\rightrightarrows}} & A \otimes_R A \otimes_R \bar{A}_{n+1} \\ \downarrow \phi & & \downarrow \phi_1 & & \downarrow \phi_2 \\ A \Box_H \bar{H}_{n+1} & \subset & A \otimes \bar{H}_{n+1} & \overset{\mathrm{id}\otimes\bar{\Delta}}{\underset{\Delta_A \otimes \mathrm{id}}{\rightrightarrows}} & A \otimes H \otimes \bar{H}_{n+1} \end{array}$$

where $i(\bar{a}) := 1 \otimes \bar{a}$, $i_1(x) := x \otimes 1$, $i_2(x) := 1 \otimes x$, $\phi_1(x \otimes \bar{a}) := \sum x a_0 \otimes \bar{a}_1$, $\phi_2(x \otimes y \otimes \bar{a}) := \sum x y_0 a_0 \otimes y_1 a_1 \otimes \bar{a}_2$ for all $a \in A_{n+1}$ and $x, y \in A$, and $\bar{\Delta}$ denotes the H-comodule structure map of $\bar{H}_{n+1}$.

Then i is injective and its image is the kernel of $i_1 \otimes \mathrm{id} - i_2 \otimes \mathrm{id}$, since A is faithfully flat over R. Also, the bottom sequence is exact by the definition of the cotensor product. It is easy to see that the diagram commutes. To show the bijectivity of ϕ it therefore suffices to show that ϕ_1 and ϕ_2 are bijective. We have seen in the proof of 3.3 that the Galois isomorphism induces isomorphisms

$$A \otimes_R A_n \xrightarrow{\cong} A \otimes H_n \quad \text{and} \quad A \otimes_R A_{n+1} \xrightarrow{\cong} A \otimes H_{n+1}$$

(take $C = H_n$ and H_{n+1}). Since A is right R-flat we see that the induced map $A \otimes_R \bar{A}_{n+1} \to A \otimes \bar{H}_{n+1}$ on the quotients, which is ϕ_1, is bijective.

The map ϕ_2 is bijective since it is the composition of the following isomorphisms:

$$A \otimes_R A \otimes_R \bar{A}_{n+1} \xrightarrow{\psi_1} (A \otimes H) \otimes_R \bar{A}_{n+1} \xrightarrow{\psi_2} A \otimes H \otimes \bar{H}_{n+1} \xrightarrow{\psi_3} A \otimes H \otimes \bar{H}_{n+1}.$$

The first map ψ_1 is the Galois map tensored with $\bar{A}_{n+1}$. ψ_2 is ϕ_1 tensored with H (the R-module structure on $A \otimes H$ is given by the left R-module structure on A), and ψ_3 is defined by $\psi_3(a \otimes x \otimes \bar{y}) = \sum a \otimes xy_1 \otimes \bar{y}_2$, since for any left H-comodule V $(= \bar{H}_{n+1}$ in $\psi_3)$,

$$H \otimes V \to H \otimes V, \quad x \otimes v \mapsto \sum xv_{-1} \otimes v_0,$$

is bijective with inverse $x \otimes v \mapsto \sum xS(v_{-1}) \otimes v_0$. ∎

To conclude from 3.5 that $I(A_{n+1}/A_n) = (A_{n+1}/A_n)I$ for ideals I in R which satisfy $IA_0 = A_0 I$ we need the following technical lemma.

Recall that a left C-comodule W is coflat if the functor $V \mapsto V \square_C W$ on right C-comodules is exact. If C is cosemisimple, i.e. $C = C_0$, then any C-comodule is coflat.

LEMMA 3.6: *Let C be a coalgebra, R an algebra and X a left R-module and a right C-comodule such that the comodule structure map $X \to X \otimes C$ is R-linear where $X \otimes C$ is an R-module via X. Then for any left C-comodule Y which is C-coflat and any ideal I in R*

$$I(X \square_C Y) = (IX) \square_C Y$$

as subsets in $X \otimes Y$.

Proof: By the assumption on X, $I \otimes_R X$ is a right C-comodule via the comodule structure of X, $X \square_C Y$ is a left R-module via multiplication on X, and IX is a sub C-comodule of X. Since Y is C-coflat, the natural map

$$\alpha\colon I \otimes_R (X \square_C Y) \to (I \otimes_R X) \square_C Y, \quad \alpha(r \otimes \sum x_i \otimes y_i) = \sum r \otimes x_i \otimes y_i$$

for $r \in R$, $\sum x_i \otimes y_i \in X\Box_C Y$, is bijective. The multiplication map $I\otimes_R X \to IX$ is surjective and induces a surjective map

$$\beta\colon (I\otimes_R X)\Box_C Y \to (IX)\Box_C Y$$

since Y is C-coflat. Hence $\beta\alpha$ is surjective. Let ι be the inclusion map of $(IX)\Box_C Y$ into $X\otimes Y$. Then the image of $\iota\beta\alpha$ is $(IX)\Box_C Y$ since $\beta\alpha$ is surjective. But by definition of α and β, $Im(\iota\beta\alpha) = I(X\Box_C Y)$. This proves the claim. ∎

We can now show the main result.

THEOREM 3.7: *Let $H_0 \subset H_1 \subset \cdots \subset H_m = H$ be the coradical filtration of H and define $t := m+1$. Then for any ideal I of R,*

$$(I : H_0)^t \subset (I : H).$$

Proof: (1) We first show for any H_0-stable ideal L of R, $A_n L^{n+1} \subset LA$ for all $0 \le n \le m$. For $n = 0$, $A_0 L \subset LA$ follows from the definition of H_0-stable ideals.

Since $LA_0 = A_0 L$ we know from 3.6 and its version for right R-modules that

$$L(A_0\Box_{H_0}\bar{H}_{n+1}) = (LA_0)\Box_{H_0}\bar{H}_{n+1} = (A_0 L)\Box_{H_0}\bar{H}_{n+1} = (A_0\Box_{H_0}\bar{H}_{n+1})L.$$

Hence by 3.5, $L\bar{A}_{n+1} = \bar{A}_{n+1}L$. In particular, $A_{n+1}L \subset LA + A_n$ for all n. Multiplying with L^{n+1} from the right gives $A_{n+1}L^{n+2} \subset LA + A_n L^{n+1}$. Hence the claim follows by induction.

(2) To prove the theorem, apply (1) to $L := (I : H_0)$ and $n = m$. Hence $AL^{m+1} \subset LA$ and also $AL^{m+1}A \subset LA$. Therefore,

$$L^t \subset (AL^tA)\cap R \subset (LA)\cap R = L$$

using 1.3(2). By 1.3(1), $(AL^tA)\cap R$ is an H-stable ideal in R. Hence $L^t \subset (L : H) = ((I : H_0) : H) \subset (I : H)$. ∎

COROLLARY 3.8: *f: H_0-Spec$(R) \to H$-Spec(R), $f(I) := (I : H)$, is bijective.*

Proof: By 3.4 f is surjective. To show injectivity let $I, J \in H_0$-Spec(R) such that $(I : H) = (J : H)$. Since I is H_0-stable, $(I : H_0) = I$ and, by 3.7, $I^t \subset (I : H) = (J : H) = J$. Hence $I \subset J$ since J is H_0-prime. In the same way we get $J \subset I$. ∎

COROLLARY 3.9: *Let $Q_1, Q_2 \in$ Spec(R). Then*

$$(Q_1 : H) = (Q_2 : H) \iff (Q_1 : H_0) = (Q_2 : H_0).$$

Proof: This follows from 3.8 since $\mathrm{Spec}(R) \to H\text{-}\mathrm{Spec}(R)$, $Q \mapsto (Q : H)$, factors through $H_0\text{-}\mathrm{Spec}(R)$. ∎

Finally we obtain the following refinement of 2.4.

COROLLARY 3.10: *Define equivalence relations on* Spec(A) *and* Spec(R) *by*

$$P_1 \approx_A P_2 \iff (P_1 : (H^*)_0) = (P_2 : (H^*)_0),$$
$$Q_1 \approx_R Q_2 \iff (Q_1 : H_0) = (Q_2 : H_0),$$

for all $P_1, P_2 \in \mathrm{Spec}(A)$ *and* $Q_1, Q_2 \in \mathrm{Spec}(R)$. *Then*

$$\mathrm{Spec}(A)/\approx_A \to \mathrm{Spec}(R)/\approx_R, \quad [P] \mapsto [Q]$$

where P *lies over* Q, *is bijective.*

Proof: Follows from 2.4 and 3.9. ∎

Let $k[H_0]$ be the subalgebra of H generated by the coradical of H. Then $k[H_0]$ is a sub Hopf algebra of H and $R \subset A(k[H_0])$ is faithfully flat $k[H_0]$-Galois by [S1, 3.11, 2]. Clearly 3.8 and 3.9 also hold when H_0 is replaced by $k[H_0]$. In a sense, this reduces the study of H-Spec(R), $R \subset A$ a faithfully flat H-Galois extension, to the case when H as an algebra is generated by its coradical.

4. The Krull relations for $R \subset A$

In this section we define versions of the usual Krull relations for our Galois extensions $R \subset A$, and show that their verification can be reduced to the special case of smash products. Recall from Definition 2.3(1) that $P \in \mathrm{Spec}(A)$ lies over $Q \in \mathrm{Spec}(R)$ if and only if $(Q : H) = P \cap R$. It is clear from Lemma 2.2 that any $P \in \mathrm{Spec}(A)$ lies over some $Q \in \mathrm{Spec}(R)$; conversely for any $Q \in \mathrm{Spec}(R)$, there exists some $P \in \mathrm{Spec}(A)$ such that P lies over Q.

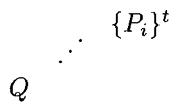
As in [P, 16.6] we may use diagrams to represent some of the Krull relations. Thus, for example, the diagram in 4.1(3) means that given $Q_1 \supset Q_2$ in Spec(R) and $P_2 \in \mathrm{Spec}(A)$ which lies over Q_2, there exists some $P_1 \in \mathrm{Spec}(A)$ such that $P_1 \supset P_2$ and P_1 lies over Q_1.

It will be convenient for us to divide the relations into "basic" Krull relations and their "duals". That in fact they are dual will be seen in Theorem 4.3.

Definition 4.1 (The basic Krull relations):

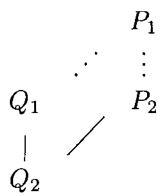
(1) H has t-**lying over** (t-LO) if for all faithfully flat H-Galois extensions $R \subset A$ and any $Q \in \mathrm{Spec}(R)$, there exist $P_1, \ldots, P_n \in \mathrm{Spec}(A)$, where $n \le \dim H$,

such that all P_i lie over Q, and such that $(\bigcap_{i=1}^{n} P_i)^t \subset (Q : H)A$.



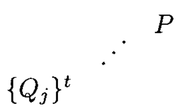
(2) H has **incomparability** (INC) if for all faithfully flat H-Galois extensions $R \subset A$ and any $P_2 \subset P_1$ in Spec(A) with $P_2 \neq P_1$, then $P_2 \cap R \neq P_1 \cap R$.

(3) H has **going up** (GU) if for all faithfully flat H-Galois extensions $R \subset A$,



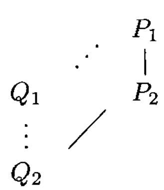
Definition 4.1′ (The dual Krull relations):

(1)′ H has ***t*-co-lying over** (t-coLO) if for all faithfully flat H-Galois extensions $R \subset A$, and any $P \in$ Spec(A), there exist $Q_1, \ldots, Q_m \in$ Spec(R), where $m \leq \dim H$, such that P lies over all Q_j, and such that $(\bigcap_{j=1}^{m} Q_j)^t \subset P \cap R$.



(2)′ H has **co-incomparability** (coINC) if for all faithfully flat H-Galois extensions $R \subset A$ and any $Q_2 \subset Q_1$ in Spec(R), with $Q_2 \neq Q_1$, then $(Q_2 : H) \neq (Q_1 : H)$.

(3)′ H has **co-going up** (coGU) if for all faithfully flat H-Galois extensions $R \subset A$,



To show that the "dual" relations are actually dual, we need a consequence of Theorem 1.4 for prime ideals.

LEMMA 4.2:

(1) *The bijection ϕ of Theorem 1.4 induces the following commutative diagrams*

on Spec*:*

$$\begin{array}{ccc} \mathrm{Spec}(R) & \xrightarrow{\phi} & \mathrm{Spec}(A\#H^*) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g'} \\ H\text{-}\mathrm{Spec}(R) & \xrightarrow{\Phi} & H^*\text{-}\mathrm{Spec}(A) \end{array}$$

where $f\colon Q \mapsto (Q : H)$, $g'\colon Q' \mapsto Q' \cap A$, *and* $\Phi\colon I \mapsto IA$. *The maps* f *and* g' *are surjective and* ϕ *and* Φ *are bijective.*

(2) *For* $P \in \mathrm{Spec}(A)$ *and* $Q \in \mathrm{Spec}(R)$, P *lies over* Q *if and only if* $\phi(Q)$ *lies over* P *(in the extension* $A \subset A\#H^*$*).*

Proof: (1) f is well-defined and surjective by Lemma 2.2(2), and g' is well-defined and surjective by 2.2(3) applied to the extension $A \subset A\#H^*$. It was also shown in 2.2(1) that Φ is bijective. The commutativity of the diagram now follows from 1.4, as the only change is in the lower right corner. However, as noted in 1.4, $(Q : H)A = Q' \cap A$.

(2) By Lemma 1.3(2), $(Q : H) = P \cap R \iff (Q : H)A = (P \cap R)A$. But by Theorem 1.4 (or (1) above), $(Q : H)A = \phi(Q) \cap A$, and by Lemma 1.3(1), $(P \cap R)A = (P : H^*)$. Thus $(Q : H) = P \cap R$ if and only if $\phi(Q) \cap A = (P : H^*)$. ∎

THEOREM 4.3: *For each of the Krull relations in* 4.1 *and* 4.1′, H^* *has a basic Krull relation if and only if* H *has its dual relation. That is,*

(1) H^* *has t-LO* $\iff$ H *has t-coLO,*
(2) H^* *has INC* $\iff$ H *has coINC,*
(3) H^* *has GU* $\iff$ H *has coGU.*

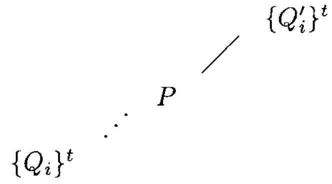
Proof: (1) ($\Rightarrow$) Assume that H^* has t-LO, and consider a faithfully flat H-Galois extension $R \subset A$. Then $A \subset A\#H^*$ is an H^*-Galois extension, and so has t-LO. Assume we are given $P \in \mathrm{Spec}(A)$. By t-LO in $A \subset A\#H^*$, there exist $Q'_1, \ldots, Q'_n \in \mathrm{Spec}(A\#H^*)$, with all Q'_i lying over P, such that

$$(\bigcap_{i=1}^{n} Q'_i)^t \subset (P : H^*)(A\#H^*) = (P : H^*)\#H^*.$$

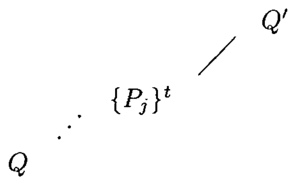
Define $Q_i := \phi^{-1}(Q'_i) \in \mathrm{Spec}(R)$; by Lemma 4.2(2), P lies over every Q_i since every Q'_i lies over P. Moreover, since ϕ preserves products and intersections, $(\bigcap_i Q_i)^t = \phi^{-1}((\bigcap_i Q'_i)^t)$. Now $I' = (P : H^*)\#H^* \in H\text{-}\mathrm{Spec}(A\#H^*)$; since $(P : H^*) = (P \cap R)A$ by Lemma 1.3(1), $\phi^{-1}(I') = P \cap R$ by Theorem 1.4. Thus

$$(\bigcap_i Q_i)^t \subset P \cap R$$

and H has t-coLO. We can represent this in the diagram



($\Leftarrow$) Conversely, assume that H has t-coLO, and consider a faithfully flat H^*-Galois extension $R \subset A$. Then $A \subset A\#H$ is H-Galois. Given a prime $Q \in \operatorname{Spec}(R)$, then $Q' = \phi(Q) \in \operatorname{Spec}(A\#H)$. Since H has t-coLO, there exist $P_1, \ldots, P_m \in \operatorname{Spec}(A)$, such that Q' lies over all P_i, and so that $(\bigcap_{j=1}^m P_j)^t \subset Q' \cap A$. But by 4.2(3), we know that each P_i lies over Q, and by Theorem 1.4, $Q' \cap A = (Q : H)A$. Thus H^* has t-LO. We express this in the diagram

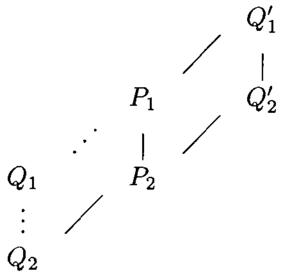


(2) ($\Rightarrow$) Assume that H^* has INC and let $R \subset A$ be a faithfully flat H-Galois extension; then $A \subset A\#H^*$ is H^*-Galois. Assume $Q_2 \subset Q_1$ in $\operatorname{Spec}(R)$, with $Q_1 \neq Q_2$, and let $Q'_i := \phi(Q_i) \in \operatorname{Spec}(A\#H^*)$. Then $Q'_2 \subset Q'_1$ and $Q'_1 \neq Q'_2$. Since H^* has INC, $Q'_2 \cap A \neq Q'_1 \cap A$. As noted in Theorem 1.4, $Q'_i \cap A = (Q_i : H)A$. Thus $(Q_2 : H)A \neq (Q_1 : H)A$. But then by Lemma 1.3(2), $(Q_2 : H) \neq (Q_1 : H)$, and so H has coINC.

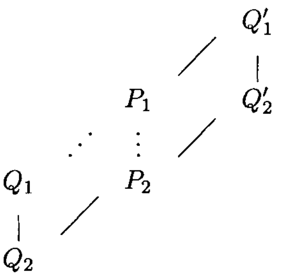
($\Leftarrow$) Assume that H has coINC, and let $R \subset A$ be a faithfully flat H^*-Galois extension; then $A \subset A\#H$ is H-Galois. Assume $P_2 \subset P_1$ in $\operatorname{Spec}(A)$ with $P_1 \neq P_2$; then $(P_1 : H) \neq (P_2 : H)$ since H has coINC. By Lemma 1.3(1), $P_i \cap R = (P_i : H) \cap R$ and $(P_i \cap R)A = (P_i : H)$. Thus $P_2 \cap R \neq P_1 \cap R$, and H^* has INC.

(3) ($\Rightarrow$) Assume that H^* has GU and let $R \subset A$ be a faithfully flat H-Galois extension; then $A \subset A\#H^*$ is H^*-Galois. Assume there are $P_2 \subset P_1$ in $\operatorname{Spec}(A)$ and $Q_2 \in \operatorname{Spec}(R)$ with P_2 lying over Q_2. Let $Q'_2 := \phi(Q_2) \in \operatorname{Spec}(A\#H^*)$;$Q'_2$ lies over P_2 by 4.3(2). Since H^* has GU, there exists $Q'_1 \in \operatorname{Spec}(A\#H^*)$ such that $Q'_2 \subset Q'_1$ and Q'_1 lies over P_1. Again by 4.3(3), P_1 lies over $\phi^{-1}(Q'_1)$; setting

$Q_1 := \phi^{-1}(Q_1')$, we see H has coGU. We express this in the diagram



($\Leftarrow$) Assume that H has coGU and let $R \subset A$ be a faithfully flat H^*-Galois extension; then $A \subset A\#H$ is H-Galois. Assume that $Q_2 \subset Q_1$ in Spec(R) and $P_2 \in$ Spec(A) with P_2 lying over Q_2 and set $Q_i' := \Phi(Q_i)$; then $Q_2' \subset Q_1'$ and Q_2' lies over P_2 by the lemma. Thus coGU for H gives some $P_1 \in$ Spec(A) with $P_1 \supset P_2$ and Q_1' lying over P_1. But then P_1 lies over Q_1 and so H^* has GU. We express this in the diagram



∎

We next show that in order to verify the Krull relations for H, we may assume that the faithfully flat H-Galois extension $R \subset A$ is of a more special form; that is, R is H-prime . Before doing this we require some facts about prime ideals and passing to quotient rings. Recall from Lemma 1.6 that for any H-stable ideal I of R, $IA = AI$ is an ideal of A, and for $\bar{R} := R/I$, $\bar{A} := A/IA$, $\bar{R} \subset \bar{A}$ is a faithfully flat H-Galois extension. Moreover, as noted in Remark 1.7, images (resp. preimages) of H-stable ideals in R ($\bar{R}$) are H-stable .

LEMMA 4.4: *Let $R \subset A$ be faithfully flat H-Galois, let I be an H-stable ideal of R, and let $\bar{R} \subset \bar{A}$ be as above. Then*

(1) *$I \in H$-Spec(R) if and only if $\bar{R} = R/I$ is an H-prime ring.*
(2) *If $P \in$ Spec(A) with $IA \subset P$, then $\bar{P} \cap \bar{R} = \{\bar{0}\} \iff P \cap R = I$.*
(3) *If $Q \in$ Spec(R) with $I \subset Q$, then $(\bar{Q} : H) = \{\bar{0}\} \iff (Q : H) = I$.*

Proof: (1) This is clear by the remarks before the lemma about H-stable ideals.

(2) ($\Rightarrow$) Assume $\bar{P} \cap \bar{R} = \{\bar{0}\}$; then $\overline{P \cap R} = \{\bar{0}\}$, and so $P \cap R \subset IA \cap R = I$. Thus $P \cap R \subset I$. Conversely $IA \subset P$ implies $I \subset P \cap R$.

($\Leftarrow$) Assume $P \cap R = I$, and choose $p \in P$, $r \in R$ such that $\bar{p} = \bar{r} \in \bar{P} \cap \bar{R}$. Then $p - r \in IA \subset P$, and so $r \in P \cap R = I$. Then $\bar{r} = \bar{0}$; that is, $\bar{P} \cap \bar{R} = \{\bar{0}\}$.

(3) ($\Rightarrow$) Since $I \subset Q$, $I \subset (Q : H)$ are H-stable ideals of R. Thus $\overline{(Q : H)}$ is H-stable in $\bar{R}$, and so $\overline{(Q : H)} \subset (\bar{Q} : H)$. Thus $(\bar{Q} : H) = \{\bar{0}\}$ implies $\overline{(Q : H)} = \{\bar{0}\}$, and so $(Q : H) \subset I$. Thus $(Q : H) = I$.

($\Leftarrow$) Assume $(Q : H) = I$, and let J be the inverse image in R of $(\bar{Q} : H)$. Then since $(\bar{Q} : H)$ is H-stable , J is H-stable , and $J \subset Q$ since $(\bar{Q} : H) \subset \bar{Q}$. Thus $J \subset (Q : H)$, and so $\bar{J} \subset \overline{(Q : H)} = \{\bar{0}\}$. Thus $(\bar{Q} : H) = \{\bar{0}\}$. ∎

PROPOSITION 4.5: *Consider extensions $R \subset A$ satisfying*

(*) *$R \subset A$ is faithfully flat H-Galois and R is H-prime.*

(1) *H has t-LO if and only if for all extensions (*) there exist $P_1, \ldots, P_n \in \mathrm{Spec}(A)$, with $n \le \dim H$, such that $P_i \cap R = \{0\}$ for all i and $(\bigcap_{i=1}^n P_i)^t = \{0\}$.*

(2) *H has INC if and only if for all extensions (*) and any $P \in \mathrm{Spec}(A)$, $P \cap R = \{0\}$ implies P is a minimal prime of A.*

(3) *H has GU if and only if for all extensions (*), the diagram for GU in 4.1(3) holds when $(Q_2 : H) = P_2 \cap R = \{0\}$.*

(1)′ *H has t-coLO if and only if for all extensions (*), there exist $Q_1, \ldots, Q_m \in \mathrm{Spec}(R)$, with $m \le \dim H$, such that $(Q_j : H) = \{0\}$ for all j and $(\bigcap_{j=1}^m Q_j)^t = \{0\}$.*

(2)′ *H has coINC if and only if for all extensions (*) and any $Q \in \mathrm{Spec}(R)$, $(Q : H) = \{0\}$ implies Q is a minimal prime of R.*

(3)′ *H has coGU if and only if for all extensions (*), the diagram for coGU in 4.1′(3)′ holds when $(Q_2 : H) = P_2 \cap R = \{0\}$.*

Proof: (1) ($\Rightarrow$) This is the special case of t-LO when $Q = \{0\}$, so that $I = (Q : H) = \{0\}$ is H-prime .

($\Leftarrow$) Assume we are given some $Q \in \mathrm{Spec}(R)$; then $I = (Q : H) \in H\text{-}\mathrm{Spec}(R)$. Let $\bar{R}$ and $\bar{A}$ be as in Lemma 4.5. Since $\bar{R}$ is H-prime , there exist $\bar{P}_1, \ldots, \bar{P}_n \in \mathrm{Spec}(\bar{A})$, with $n \le \dim H$, such that $\bar{P}_i \cap \bar{R} = \{\bar{0}\}$ for all i and $(\bigcap_i \bar{P}_i)^t = (\bar{0})$. Let P_i be the inverse image of $\bar{P}_i$ in A; then each $P_i \in \mathrm{Spec}(A)$, and $P_i \cap R = I$ by Lemma 4.5(2). Thus each P_i lies over Q, since $I = (Q : H)$. Moreover $(\bigcap_i P_i)^t \subset IA$.

(2) ($\Rightarrow$) This is a special case of INC. For if $P \cap R = \{0\}$ and P is not minimal, then $P_2 \subsetneq P$ for some $P_2 \in \operatorname{Spec}(A)$. But then $P_2 \cap R \neq P \cap R$, a contradiction since $P_2 \cap R \subset P \cap R = \{0\}$.

($\Leftarrow$) Assume we are given $P_2 \subsetneq P_1$ in $\operatorname{Spec}(A)$ and let $I = P_2 \cap R$; $I \in H\text{-}\operatorname{Spec}(R)$ by Lemma 2.2(3). Passing to $\bar{R}$ and $\bar{A}$ as in (1), $\bar{R}$ is H-prime and $\bar{P}_2 \subsetneq \bar{P}_1$ in $\operatorname{Spec}(A)$. By Lemma 4.5, $\bar{P}_2 \cap \bar{R} = (\bar{0})$. If $P_1 \cap R = P_2 \cap R$, then also $\bar{P}_1 \cap \bar{R} = (\bar{0})$. By hypothesis this implies $\bar{P}_1$ is minimal, a contradiction. Thus $P_1 \cap R \neq P_2 \cap R$ and H has INC.

(3) This is proved by going to $\bar{R} \subset \bar{A}$ as above.

(1)′ Although this is just the dual of (1), we give the details.

($\Rightarrow$) This is the special case when $P \in \operatorname{Spec}(A)$ such that $P \cap R = 0$; such a P exists by Lemma 2.2(3).

($\Leftarrow$) Assume we are given some $P \in \operatorname{Spec}(A)$; then $I = P \cap R \in H\text{-}\operatorname{Spec}(R)$. Passing to $\bar{R}$ and $\bar{A}$ as before, we may assume $\bar{I} = \{\bar{0}\}$ and $\bar{R}$ is H-prime . Thus by hypothesis, there exist $\bar{Q}_1, \ldots, \bar{Q}_m \in \operatorname{Spec}(\bar{R})$ such that $(\bar{Q}_j : H) = \{\bar{0}\}$ for all j and such that $(\bigcap_j \bar{Q}_j)^t = (\bar{0})$. Let $Q_1, \ldots, Q_m \in \operatorname{Spec}(R)$ be the inverse images of $\bar{Q}_1, \ldots, \bar{Q}_m$ respectively; by Lemma 4.5(3), $(Q_j : H) = I = P \cap R$ and thus P lies over each Q_j. Moreover $(\bigcap_j Q_j)^t \subset I = P \cap R$. Thus H has t-coLO.

(2)′ ($\Rightarrow$) This is a special case of coINC. For if $(Q : H) = \{0\}$ and Q is not minimal, then $Q_2 \subsetneq Q$ for some $Q_2 \in \operatorname{Spec}(R)$. But then $(Q_2 : H) \neq (Q : H)$, a contradiction since $(Q_2 : H) = \{0\}$.

($\Leftarrow$) Assume that $Q_2 \subsetneq Q_1$ in $\operatorname{Spec}(R)$ and let $I = (Q_2 : H)$; then $I \in H\text{-}\operatorname{Spec}(R)$ by Lemma 2.2(2). Passing to $\bar{R}$ and $\bar{A}$ again, we see that $(\bar{Q}_2 : H) = \{\bar{0}\}$ by Lemma 4.5(3). If $(Q_1 : H) = (Q_2 : H)$, then also $(\bar{Q}_1 : H) = \{\bar{0}\}$ and so by our hypothesis, $\bar{Q}_1$ is a minimal prime of $\bar{R}$. But $I \subset Q_2 \subsetneq Q_1$ implies $\bar{Q}_2 \subsetneq \bar{Q}_1$, in $\bar{R}$, a contradiction. Thus $(Q_1 : H) \neq (Q_2 : H)$ and H has coINC.

(3)′ This is proved routinely by passing to $\bar{R} \subset \bar{A}$. ∎

In fact it is sufficient to verify the Krull relations for H in a much more special situation.

COROLLARY 4.6: *For any of the Krull relations in Definitions* 4.1 *and* 4.1′, *H satisfies the given relation if and only if it satisfies the relation for H-Galois extensions of the form $A = R \# H$, where R is an H-module algebra.*

Moreover, one may assume in addition that R is H-prime and that the Krull relations have the special forms in Proposition 4.5.

Proof: First note that in the proof of Theorem 4.3, in each case only the fact that H (or H^*) satisfied the assumed property for the case $A \subset A \# H$ (or $A \subset A \# H^*$)

was needed; that is, for the case when the Galois extensions is an ordinary smash product. This observation implies the first part of the corollary, for if H satisfies the property for smash products, then H^* satisfies its dual, and so H satisfies the property in general, using Theorem 4.3 and its proof.

To see the second part, we may pass to quotients as in Proposition 4.5. One only needs the additional fact, as noted in Remark 1.7(2), that if $A = R\#H$, then $\bar{A} \cong \bar{R}\#H$, again a smash product. ∎

We can now connect our definition of lying over to the more usual one involving minimal primes.

COROLLARY 4.7: *Let $R \subset A$ be faithfully flat H-Galois. Assume $P \in \mathrm{Spec}(A)$ and $Q \in \mathrm{Spec}(R)$ as usual.*

(1) *If H has t-LO and INC, then*

$$P \text{ lies over } Q \iff P \text{ is minimal over } (Q:H)A.$$

Assume in addition that R is H-prime . Then P is minimal in $\mathrm{Spec}(A)$ if and only if $P \cap R = \{0\}$. Moreover A has $n \leq \dim H$ minimal primes, call them $P_1, \dots P_n$; if $N = \bigcap_{i=1}^n P_i$, then $N^t = \{0\}$ and N is the largest nilpotent ideal of A.

(2) *If H has t-coLO and coINC, then*

$$P \text{ lies over } Q \iff Q \text{ is minimal over } P \cap R.$$

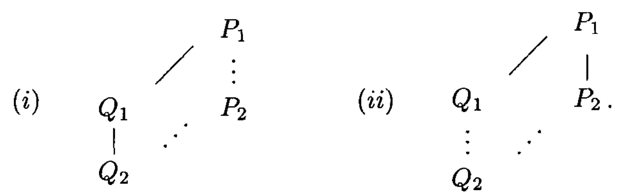
Assume also that R is H-prime . Then Q is minimal in $\mathrm{Spec}(R)$ if and only if $(Q : H) = \{0\}$. Moreover R has $m \leq \dim H$ minimal primes, call them $Q_1, \dots, Q_m$; if $N = \bigcap_{j=1}^m Q_j$, then $N^t = \{0\}$ and N is the largest nilpotent ideal of R.

Proof: (1) First assume that P is minimal over $(Q : H)A$. If H has t-LO, then there exist $P_1, \dots, P_n \in \mathrm{Spec}(A)$, with all P_i lying over Q, such that $(\bigcap_{i=1}^n P_i)^t \subset (Q : H)A \subset P$. Since P is prime, $P_j \subset P$ for some j; since P is minimal, $P = P_j$. Thus P lies over Q. Conversely assume that P lies over Q, and say that there is some $P_2 \in \mathrm{Spec}(A)$ such that $(Q : H)A \subset P_2 \subset P$. If H has INC and $P_2 \neq P$ then $P_2 \cap R \neq P \cap R$; but $(Q : H) = P_2 \cap R = P \cap R$ since P lies over Q, a contradiction. Thus $P_2 = P$ and P is minimal.

When R is H-prime , $I = \{0\}$ is an H-prime ideal, and we apply the above fact to the case when $(Q : H) = \{0\}$ to see that P is minimal in $\mathrm{Spec}(A)$ if and only if $P \cap R = (Q : H) = \{0\}$. Now use the formulation of t-LO in Proposition 4.5(1) to see that A has primes $P_1, \dots, P_n$ with $P_i \cap R = \{0\}$ (which are necessarily minimal by the above) such that if $N = \bigcap_{i=1}^n P_i$, $N^t = \{0\}$. Since for any other $P \in \operatorname{Spec}(A)$, $(\prod_i P_i)^t \subset P$, and so $P_i \subset P$ for some i, the set $\{P_1, \dots, P_n\}$ are all the minimal primes of A. Moreover since A/N is semiprime, N is the largest nilpotent ideal of A.

(2) This argument is similar to (1). If Q is minimal over $P \cap R$, it follows from t-coLO that P lies over Q. Conversely if P lies over Q, one uses coINC to see that Q is minimal over $P \cap R$. Thus P lies over Q if and only if Q is minimal over $P \cap R$, and thus when R is H-prime , it follows that Q is minimal in $\operatorname{Spec}(R)$ if and only if $(Q : H) = \{0\}$. The last statement about N and the set of minimal primes now follows from the formulation of t-coLO in Proposition 4.5(1)$'$. ∎

Remark 4.8: In this section we have ignored one of the fundamental Krull relations, namely going down (GD); this is because it is in fact only a weaker version of t-LO. To see this, one can make the definitions (analogous to going up) as follows: H has GD (resp. coGD) if for all faithfully flat H-Galois extensions $R \subset A$, the diagram (i) (resp. (ii)) holds:



As we showed for the other Krull relations, GD and coGD are dual, and it suffices to check them for smash products $A = R\#H$ in which R is H-prime .

We now compare these definitions with t-lying over:

SUBLEMMA: *If H has t-LO (respectively, t-coLO), then H has GD (resp., coGD).*

Proof: By duality it suffices to show that t-LO implies GD. Thus we assume that $Q_2 \subset Q_1$ in $\operatorname{Spec}(R)$ and that P_1 lies over Q_1. By t-LO, there exist $P_{2_1}, \dots P_{2_m} \in \operatorname{Spec}(A)$ such that all P_{2_i} lie over Q_2 and such that $(\bigcap_{i=1}^m P_{2_i})^t \subset (Q_2 : H)A$. But $(Q_2 : H)A \subset (Q_1 : H)A = (P_1 \cap R)A \subset P_1$. Thus for some j, $P_{2_j} \subset P_1$, and P_{2_j} lies over Q_2. By setting $P_2 = P_{2_j}$, the result is proved. ∎

Example 4.9: Let $H = kG$ be a group algebra. By Ulbrich's theorem, $R \subset A$ is H-Galois if and only if A is strongly G-graded with $R = A_1$, that is $A_g A_h = A_{gh}$ for all $g, h \in G$, where $A = \bigoplus_{g \in G} A_g$. Thus in this case faithful flatness is automatic, since R is a direct summand of A.

We claim that H satisfies all of the Krull relations in 4.1 and 4.1′. First, by Corollary 4.6, it suffices to show that they hold when R is an H-module algebra and $A = R\#H$, that is when there is an ordinary action of G on R. But now the "dual" Krull relations t-coLO, coINC, and coGU are almost trivial. For, if $I \in G\text{-Spec}(R)$ then there exists $Q \in \mathrm{Spec}(R)$ such that $I = \bigcap_{x\in G}(x \cdot Q)$; conversely given $Q \in \mathrm{Spec}(R)$, then $(Q : G) = \bigcap_{x\in G}(x \cdot Q)$ is G-prime. Thus kG satisfies 1-coLO. To see coINC, note that $(Q : G) = (Q' : G)$ if and only if $Q' = x \cdot Q$ for some $x \in G$. Finally coGU holds, for if $P_1 \supset P_2$ in $\mathrm{Spec}(A)$ and P_2 lies over Q_2, then $P_1 \cap R = \bigcap_x x \cdot Q$ for some $Q \in \mathrm{Spec}(R)$ since $P_1 \cap R$ is G-prime. Thus $P_2 \cap R = \bigcap_x x \cdot Q_2 \subset P_1 \cap R \subset Q$, and so for some $x \in G$, $x \cdot Q_2 \subset Q$. But then $Q_2 \subset x^{-1} \cdot Q$ and we may set $Q_1 := x^{-1} \cdot Q$.

The fact that H satisfies t-LO, INC, and GU follows from the rather non-trivial work of Lorenz and Passman [P, 16.6], where these relations are shown for crossed products $A = R * G = R\#_\tau kG$; again, this suffices by 4.6.

Thus all of the Krull relations hold for strongly graded rings; dualizing, all of the Krull relations hold for a faithfully flat $(kG)^*$-Galois extension $R \subset A$ (that is, there is a G-action on A).

Remark 4.10: We now consider the more general case when H is finite-dimensional and pointed. Then the question as to whether H has t-LO or INC is open. But H has m-coLO for some $m \leq \dim H$ depending on the coradical filtration of H, coINC, coGU and GU. Moreover, lying over in the classical sense does hold; that is, for any $Q \in \mathrm{Spec}(R)$, there exists $P \in \mathrm{Spec}(A)$ such that Q is minimal over $P \cap R$.

Proof: Again by Corollary 4.6 it suffices to consider extensions $R \subset A = R\#H$ where R is an H-module algebra. Then GU is shown in [CRW, 3.6]. Using Chin's Lemma 2.5 the other Krull relations follow easily as in the previous example for group algebras. To see m-coLO, let $P \in \mathrm{Spec}(A)$. By 2.2, $P \cap R = (Q : H)$ for some $Q \in \mathrm{Spec}(R)$. By 2.5(1),

$$\left(\bigcap_{x\in G}(x \cdot Q)\right)^m \subset (Q : H).$$

This shows m-coLO. As in 4.9, coINC holds since for prime ideals $Q_1, Q_2 \in \mathrm{Spec}(R)$, $(Q_1 : H) = (Q_2 : H)$ if and only if $Q_2 = x \cdot Q$ for some $x \in G(H)$ by 2.5(2). In the same way one shows coGU (see [Ch90, 3.6]). Finally, we see that classical lying over holds. If $Q \in \mathrm{Spec}(R)$, then Lemma 2.2 gives some $P \in \mathrm{Spec}(A)$ such that $P \cap R = (Q : H)$. But now coINC implies that Q is minimal over $P \cap R$. ∎

5. Comodule algebras with a total integral (or, module algebras with a surjective trace)

In this section we show that for many of the Krull relations of Section 4, our assumption that $R \subset A$ be faithfully flat H-Galois can be considerably weakened. We consider here H-comodule algebras A, with $R = A^{\mathrm{co}H}$, such that A has a **total integral** for H in the sense of Doi: that is, there exists a unital right H-comodule map $\gamma\colon H \to A$. Equivalently, A is an injective H-comodule [D]. Such a map γ always exists if $R \subset A$ is faithfully flat H-Galois [D, 1.6], [S1, Th. I].

Since the arguments we use generalize those for the action of a finite group in [M81] and for group graded rings in [CM], [MSm], we will work here with actions of H rather than coactions. Thus, assume that H is finite-dimensional and let A be an H-module algebra with invariants $R = A^H$. Choose $t \in \int_H^l$; then $\hat{t}\colon A \to R$ given by $\hat{t}(a) = t \cdot a$ is a **trace map.** $\hat{t}$ is **surjective** if $\hat{t}(A) = R$; equivalently there exists $c \in A$ such that $t \cdot c = 1$.

That this notion is equivalent to the one above was shown by Doi and by Cohen and Fischman; see [M, 4.3.9]. In particular the trace map is always surjective if H is semisimple.

LEMMA 5.1: *Let A be a left H-module algebra, and consider A as a right H^*-comodule algebra. Then $\hat{t}\colon A \to A^H$ is surjective if and only if there exists a total integral $\gamma\colon H^* \to A$.*

Since our extension $R \subset A$ is not Galois, we do not have a correspondence between ideals as in Lemma 1.3. Instead, the role of "H-prime " ideals of R will be played by the set of ideals

$$\{P \cap R \mid P \in \mathrm{Spec}(A)\}$$

and our notion of P lying over $\mathfrak{q}$ in 2.3 is replaced by $\mathfrak{q}$ being minimal over $P \cap R$, for $P \in \mathrm{Spec}(A)$ and $\mathfrak{q} \in \mathrm{Spec}(R)$. Morita equivalence of $A\#H$ with R will be replaced with a weaker correspondence, as follows. We need a classical lemma.

LEMMA 5.2: *Let S be a ring and $0 \neq e = e^2 \in S$. Let $\varphi\colon S \to eSe$ be given by $s \mapsto ese$, and let $\mathrm{Spec}_e(S) = \{P \in \mathrm{Spec}(S) \mid e \notin P\}$. Then:*

(1) *φ takes ideals of S to ideals of eSe, preserves intersection of ideals, and for $I, J \lhd S$, $\varphi(I)\varphi(J) \subset \varphi(IJ)$.*

(2) *φ preserves containments, and if $I \lhd S$, $P \in \mathrm{Spec}_e(S)$ with $\varphi(I) \subset \varphi(P)$, then $I \subset P$.*

(3) *φ induces a bijection $\mathrm{Spec}_e(S) \xrightarrow{\cong} \mathrm{Spec}(eSe)$.*

For a proof of (3), see [P, 17.8]; (2) is implicit in those arguments and (1) is straightforward.

In our situation, more can be said. From [M, 4.3.4], we have

LEMMA 5.3: *Le A be a left H-module algebra with surjective trace, say $t \cdot c = 1$, and let $S = A\#H$. Then $e := (1\#t)(c\#1) = tc$ is an idempotent in S and $eSe = A^H e \cong A^H$, where the last isomorphism (of algebras) is given by $ae \mapsto a$.*

The proof of 5.3 follows from the elementary computation

$$(*) \qquad e(a\#h)e = \varepsilon(h)eae = \varepsilon(h)(t \cdot (ca))e.$$

Thus when $a \in A^H$, $eae = ae$ since $t \cdot c = 1$.

LEMMA 5.4: *Let A and e be as in 5.3 and let φ: $A\#H \to A^H e$ be as in 5.2.*

(1) For any $J \lhd A$,

$$\varphi((J:H)\#H) = \varphi((J:H)) = (J \cap R)e.$$

(2) For $\mathfrak{q} \in \operatorname{Spec}(R)$ and $P \in \operatorname{Spec}(A)$,

$$\mathfrak{q} \text{ is minimal over } P \cap R \iff Q = \varphi^{-1}(\mathfrak{q}) \text{ is minimal over } I = (P:H)\#H \text{ in } A\#H$$

(here we have identified $(J \cap R)e$ with $J \cap R$ and $\mathfrak{q}e$ with $\mathfrak{q}$).

Proof: (1) $\varphi((J:H)\#H) = e((J:H)\#H)e = e(J:H)e = \varphi((J:H))$, by $(*)$, and $e(J:H)e = (t \cdot (c(J:H)))e \subset (t \cdot (J:H))e \subset (J \cap A^H)e \cong J \cap R$.

Conversely, clearly $J \cap R \subset (J:H)$, and so

$$(J \cap R)e = e(J \cap R)e = \varphi(J \cap R) \subset \varphi((J:H)).$$

(2) ($\Rightarrow$) Assume $\mathfrak{q}$ is minimal over $P \cap R$. If Q is not minimal over I, choose $Q_2 \in \operatorname{Spec}(A\#H)$ with $I \subset Q_2 \subsetneq Q$. Now $e \notin Q$ since $\varphi(Q) \neq R$; thus $Q_2, Q \in \operatorname{Spec}_e(S)$. By Lemma 5.2(3) it follows that $\varphi(I) \subset \varphi(Q_2) \subsetneq \varphi(Q)$. Setting $\mathfrak{q}_2 := \varphi(Q_2)$ and using part (1), we see $P \cap R \subset \mathfrak{q}_2 \subsetneq \mathfrak{q}$, a contradiction to the minimality of $\mathfrak{q}$. Thus Q is minimal over I.

($\Leftarrow$) Assume Q is minimal over I, but for some $\mathfrak{q}_2 \in \operatorname{Spec}(R)$, $P \cap R \subset \mathfrak{q}_2 \subsetneq \mathfrak{q}$. Let $Q_2 := \varphi^{-1}(\mathfrak{q}_2) \in \operatorname{Spec}_e(A\#H)$; then $Q_2 \subsetneq Q$ since φ is bijective on $\operatorname{Spec}_e(A\#H)$. Also by 5.2(2), $I \subset Q_2$ since Q_2 is prime and $\varphi(I) = P \cap R \subset \varphi(Q_2) = \mathfrak{q}_2$. This contradicts the minimality of Q. ∎

THEOREM 5.5: *Let A be an H-module algebra with surjective trace and let $R = A^H$. Assume that H satisfies t-LO.*

1. *For each $P \in \operatorname{Spec}(A)$, there are (only) a finite number of primes in $\operatorname{Spec}(R)$ minimal over $P \cap R$, call them $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$, and $1 \leq n \leq \dim H$. Moreover if $N = \bigcap_{i=1}^n \mathfrak{q}_i$, then $N^t \subset P \cap R$.*
2. *If also H has INC, then for each $\mathfrak{q} \in \operatorname{Spec}(R)$, there exists $P \in \operatorname{Spec}(A)$ with $\mathfrak{q}$ minimal over $P \cap R$, and if $\mathfrak{q}$ is minimal over $P' \cap R$ for some other $P' \in \operatorname{Spec}(A)$, then $P' \cap R = P \cap R$. If in addition H has t-coLO, there are only $m \leq \dim H$ such primes in $\operatorname{Spec}(A)$.*
3. *If also H has coINC, and $P_2 \subsetneq P_1$ in $\operatorname{Spec}(A)$, then $P_2 \cap R \neq P_1 \cap R$.*
4. *If also H has coGU and INC, then $R \subset A$ has "going up" in the sense that if $\mathfrak{q}_2 \subset \mathfrak{q}_1$ in $\operatorname{Spec}(R)$ and $P_2 \in \operatorname{Spec}(A)$ with $\mathfrak{q}_2$ minimal over $P_2 \cap R$, then there exists $P_1 \in \operatorname{Spec}(A)$ with $P_1 \supset P_2$ and $\mathfrak{q}_1$ minimal over $P_1 \cap R$.*

Proof: (1) We apply t-LO to the extension $A \subset A\#H$, to see that there exist $Q_1, \ldots, Q_m \in \operatorname{Spec}(A\#H)$, $m \leq \dim H$, such that $Q_i \cap A = (P : H)$ for all i; moreover, if $M = \bigcap_{i=1}^m Q_i$ then $M^t \subset (P : H)\#H$. Applying φ and Lemma 5.4(1),

$$\varphi(M)^t \subset \varphi(M^t) \subset \varphi((P : H)\#H) = P \cap R.$$

But also $\varphi(M) = \bigcap_{i=1}^m \varphi(Q_i)$. In this intersection, we may omit any $\varphi(Q_i)$ such that $e \in Q_i$, for then $\varphi(Q_i) = R$. We may also omit any $\varphi(Q_i)$ such that $\varphi(Q_j) \subset \varphi(Q_i)$ for some j. Thus by renumbering, we may write $\varphi(M) = \bigcap_{i=1}^n \varphi(Q_i)$, $n \leq m$, such that the $\{\varphi(Q_i)\}$ are proper and incomparable. Now set $\mathfrak{q}_i := \varphi^{-1}(Q_i)$ and $N = \varphi(M)$; then $N = \bigcap_{i=1}^n \mathfrak{q}_i$ and $N^t \subset P \cap R$ by the above. We note here that $e \notin N^t$, so $e \notin N$, so there is at least one proper $\mathfrak{q}_i$. If $\mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{q} \supset P \cap R$, then $\mathfrak{q} \supset N^t$ and so $\mathfrak{q} \supset \mathfrak{q}_i$, for some i. It follows that $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ is precisely the set of primes in R minimal over $P \cap R$.

(2) Let $Q = \varphi^{-1}(\mathfrak{q}) \in \operatorname{Spec}_e(A\#H)$. Then by Lemma 2.2, there exists $P \in \operatorname{Spec}(A)$ with $(P : H) = Q \cap A$. By INC and Corollary 4.7(1), Q is minimal over $(P : H)\#H$, and so by Lemma 5.4(2), $\mathfrak{q}$ is minimal over $P \cap R$. If also $\mathfrak{q}$ is minimal over $P' \cap R$, then 5.4(2) implies Q is minimal over $(P' : H)\#H$. By 4.7(1) again, $Q \cap A = (P' : H)$. But then $(P : H) = (P' : H)$ and so $P \cap R = P' \cap R$ by 5.4(1).

The above argument shows that for any $P \in \operatorname{Spec}(A)$, $\mathfrak{q}$ is minimal over $P \cap R$ if and only if $(P : H) = Q \cap A = \varphi^{-1}(\mathfrak{q}) \cap A$. But when H has t-coLO, there are only $m \leq \dim H$ such primes in $\operatorname{Spec}(A)$.

(3) Assume that $P_2 \subsetneq P_1$ in $\operatorname{Spec}(A)$. Then by coINC, $(P_2 : H) \subsetneq (P_1 : H)$, and thus in $A\#H$, $I_2 = (P_2 : H)\#H \subsetneq I_1 = (P_1 : H)\#H$. By t-LO, there are

$\{L_{2_j}\}, \{L_{1_i}\} \subset \operatorname{Spec}(A\#H)$, with at most $\dim H$ primes in each set, such that $L_{2_j} \cap A = P_2$ and $L_{1_i} \cap A = P_1$ for all i, j. Moreover, if $N_1 = \bigcap_i L_{1_i}$ and $N_2 = \bigcap_j L_{2_j}$, then ${N_1}^t \subset I_1$, and ${N_2}^t \subset I_2$. Choose L_{1_k} so that $e \notin L_{1_k}$ (this is possible as in (1), since otherwise we would have $e \in N_1$, and so $e \in \varphi({N_1}^t) \subset P_1 \cap R$, a contradiction). We may also assume that L_{1_k} is minimal over I_1, as in (1). Then

$$\Big(\bigcap_j L_{2_j}\Big)^t = {N_2}^t \subset I_2 \subsetneq I_1 \subset L_{1_k},$$

and thus for some j, $L_{2_j} \subset L_{1_k}$. In fact $L_{2_j} \neq L_{1_k}$ since they have distinct intersections with A. Since $L_{2_j}, L_{1_k} \in \operatorname{Spec}_e(A\#H)$, $\varphi(L_{2_j}) \subsetneq \varphi(L_{1_k})$. Moreover $\varphi(I_2) = P_2 \cap R \subset \varphi(L_{2_j})$. Setting $\mathfrak{q}_2 = \varphi(L_{2_j})$ and $\mathfrak{q}_1 = \varphi(L_{1_k})$, if $P_2 \cap R = P_1 \cap R$ we get

$$P_1 \cap R \subset \mathfrak{q}_2 \subsetneq \mathfrak{q}_1.$$

But by Lemma 5.4(2) and our assumption on L_{1_k}, $\mathfrak{q}_1$ is minimal over $P_1 \cap R$, a contradiction. Thus $P_2 \cap R \neq P_1 \cap R$.

(4) Assume we are given $\mathfrak{q}_2 \subset \mathfrak{q}_1$, and P_2 as in the statement of (4). Let $Q_i = \varphi^{-1}(\mathfrak{q}_i) \in \operatorname{Spec}_e(A\#H)$; then Q_2 is minimal over $(P_2 : H)\#H$ by Lemma 5.4(1), and so by Corollary 4.7(1), $(P_2 : H) = Q_2 \cap A$. By coGU, there exists $P_1 \supset P_2$ so that Q_1 lies over P_1 in our old sense, that is $(P_1 : H) = Q_1 \cap A$. By 4.7 again, this means that Q_1 is minimal over $(P_1 : H)\#H$. Again by 5.4(2), $\mathfrak{q}_1$ is minimal over P_1. ∎

We can now extend the notion of equivalent primes in R^G from [M81].

Definition 5.6: Let A be an H-module algebra, with $R = A^H$. For $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}(R)$, we say

$$\mathfrak{q}_1 \sim_R \mathfrak{q}_2 \iff \text{there exists } P \in \operatorname{Spec}(A) \text{ such that } \mathfrak{q}_1 \text{ and } \mathfrak{q}_2 \text{ are both minimal over } P \cap R.$$

In general $\sim_R$ is not an equivalence relation; however, Theorem 5.5 gives sufficient conditions for this to happen. The corollary extends [M81] for group actions and [MSm] for group gradings.

COROLLARY 5.7: *Let A be an H-module algebra with surjective trace, let $R = A^H$, and assume H has t-LO and INC. Then:*

(1) *$\sim_R$ as in Definition 5.6 is an equivalence relation on* $\operatorname{Spec}(R)$. *Each equivalence class $[\mathfrak{q}]$ contains at most* $\dim H$ *elements.*

(2) *The map*

$$f\colon \mathrm{Spec}(A) \to \mathrm{Spec}(R)/\sim_R, P \mapsto \{\mathfrak{q} \in \mathrm{Spec}(R) \mid \mathfrak{q} \textit{ is minimal over } P \cap R\}$$

is well-defined, surjective, and induces a bijection

$$\tilde{f}\colon \mathrm{Spec}(A)/\sim_H \xrightarrow{\cong} \mathrm{Spec}(R)/\sim_R$$

where $\sim_H$ *is as in Definition 2.3.*

Proof: (1) First, $\sim_R$ is reflexive since, by Theorem 5.5(2), there exists $P \in \mathrm{Spec}(R)$ such that $\mathfrak{q}$ is minimal over $P \cap R$, and $\sim_R$ is trivially symmetric. To see that it is transitive, assume $\mathfrak{q}_1 \sim_R \mathfrak{q}_2$ and $\mathfrak{q}_2 \sim_R \mathfrak{q}_3$. Then there exist $P_1, P_2 \in \mathrm{Spec}(A)$ such that $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are minimal over $P_1 \cap R$ and such that $\mathfrak{q}_2$ and $\mathfrak{q}_3$ are minimal over $P_2 \cap R$. But then $\mathfrak{q}_2$ is minimal over both $P_1 \cap R$ and $P_2 \cap R$; by 5.5(2), $P_1 \cap R = P_2 \cap R$ and so $\mathfrak{q}_1 \sim_R \mathfrak{q}_3$. The second statement follows from 5.5(1).

(2) By (1), $f(P) = [\mathfrak{q}]$, an equivalence class in $\mathrm{Spec}(R)$, except for the possibility that there are no primes $\mathfrak{q}$ minimal over $P \cap R$; however, this cannot happen by Theorem 5.5(1). Thus f is well-defined. f is surjective by 5.5(2). Now if $f(P) = f(P')$, then some $\mathfrak{q} \in \mathrm{Spec}(R)$ is minimal over both $P \cap R$ and $P' \cap R$. By 5.5(2), $P \cap R = P' \cap R$ and thus (as noted in the proof) $(P : H) = (P' : H)$. That is, $P \sim_H P'$. Conversely,

$$P \sim_H P' \;\Rightarrow\; (P : H) = (P' : H) \;\Rightarrow\; P \cap R = P' \cap R \;\Rightarrow\; f(P) = f(P').$$

Thus f induces a bijection $\tilde{f}$ on the quotient spaces. ∎

Part (2) can be considered as a generalization of Corollary 2.4 when H also has t-LO and INC, because by 4.7 "lying over" is equivalent to "minimal over $P \cap R$".

We remark that unlike the faithfully flat Galois case, not all the Krull relations hold in such extensions, even if H (and H^*) have all the relations in Definition 4.1 and 4.1′. For example, the analog of "coGU" fails for an action of $H = k\mathbb{Z}_2$ over a field of characteristic not 2, by an example of Montgomery and Small [MSm], [P, pp. 289–290]; since $H \cong H^*$, it also fails for $\mathbb{Z}_2$-graded rings.

6. Transitivity

We consider when the class of finite Hopf algebras satisfying various of the Krull relations is closed under extensions.

In this section, let H be a finite-dimensional Hopf algebra , K a normal sub Hopf algebra of H, $\bar{H} := H/HK^+$ and $\pi: H \to \bar{H}$, $\pi(h) = \bar{h}$, the quotient map. Recall that a sub Hopf algebra K of H is normal if it is stable under both adjoint actions, i.e. $\sum h_1 k S(h_2)$ and $\sum S(h_1) k h_2$ are in K for all $k \in K$ and $h \in H$. (In the finite case it suffices to assume stability under one of the adjoint actions.)

LEMMA 6.1:

(1) *$K \subset H \xrightarrow{\pi} \bar{H}$ is a strictly exact sequence of Hopf algebras [S4], i.e. H is (left and right) faithfully flat over K. Hence $H \to \bar{H}$ is conormal and $K = H^{\mathrm{co}\bar{H}}$.*

(2) *The dual sequence of Hopf algebras $\bar{H}^* \to H^* \to K^*$ is again strictly exact.*

Proof: By [NZ] H is free over K, hence faithfully flat . Then $H \to \bar{H}$ is conormal and $K = H^{\mathrm{co}\bar{H}}$ by [S4, 1.4]. Hence by duality, $\bar{H}^*$ is isomorphic to a normal sub Hopf algebra of H^*, and the dual sequence is strictly exact. ∎

We also fix a faithfully flat H-Galois extension $R \subset A$ and let

$$B := A(K) = \Delta_A^{-1}(A \otimes K).$$

Then B is a right K-comodule algebra by restricting Δ_A to B. A will be considered as a right $\bar{H}$-comodule algebra via

$$A \xrightarrow{\Delta_A} A \otimes H \xrightarrow{\mathrm{id}\otimes\pi} A \otimes \bar{H}.$$

LEMMA 6.2: *$R \subset B$ is faithfully flat K-Galois, and $B \subset A$ is faithfully flat $\bar{H}$-Galois. (This also holds for infinite-dimensional H when H is faithfully flat over K.)*

Proof: $R \subset A(K) = B$ is faithfully flat K-Galois for any sub Hopf algebra K by [S1, 3.11(2)]. $A^{\mathrm{co}\bar{H}} \subset A$ is faithfully flat $\bar{H}$-Galois by [S1, 3.10] since H is right $\bar{H}$-coflat by [NZ]. Thus it remains to show that $B = A^{\mathrm{co}\bar{H}}$. For all $a \in B$, $\sum a_0 \otimes a_1 \in A \otimes K$, hence $\sum a_0 \otimes \bar{a}_1 = a \otimes \bar{1}$ since $\bar{a}_1 = \varepsilon(a_1)\bar{1}$, and so $a \in A^{\mathrm{co}\bar{H}}$. Conversely, if $a \in A^{\mathrm{co}\bar{H}}$, then $\sum a_0 \otimes \bar{a}_1 = a \otimes \bar{1}$ and $\sum a_0 \otimes a_1 \otimes \bar{a}_2 = \sum a_0 \otimes a_1 \otimes \bar{1}$. Hence $\sum a_0 \otimes a_1 \in A \otimes H^{\mathrm{co}\bar{H}} = A \otimes K$ since $H^{\mathrm{co}\bar{H}} = K$ by 6.1. Thus $a \in B$. ∎

The next lemma is needed to prove transitivity of Krull relations from $R \subset B$ and $B \subset A$ to $R \subset A$, that is, to prove "going up" from K and $\bar{H}$ to H.

LEMMA 6.3:

(1) *H-stable ideals in R are K-stable, and for any ideal I in R, $((I : K) : H) = (I : H)$.*

(2) *For any ideal J in B, $(J : \bar{H}) \cap R = ((J \cap R) : H)$.*

Proof: (1) This is a special case of 3.3. However, here the proof follows directly from the category equivalences $\mathcal{M}_B \cong \mathcal{M}_A^{\bar{H}}$ and ${}_B\mathcal{M} \cong_A \mathcal{M}^{\bar{H}}$. For any ideal I in R these equivalences imply $(IA) \cap B = (IBA) \cap B = IB$ and $(AI) \cap B = BI$. Thus $IB = BI$ if $IA = AI$.

(2) (a) First we show that $J \cap R$ is H-stable for any $\bar{H}$-stable ideal J in B. Since J is $\bar{H}$-stable, $JA = AJ$ is an ideal in A and $JA \cap R$ is H-stable by 1.3(1). By 1.3(2), $(JA) \cap B = J$. Hence $J \cap R = (JA) \cap B \cap R = (JA) \cap R$ is H-stable .

(b) For an arbitrary ideal J in B, $(J : \bar{H})$ is $\bar{H}$-stable. Hence $(J : \bar{H}) \cap R$ is H-stable by (a). Thus

$$(J : \bar{H}) \cap R \subset ((J \cap R) : H).$$

To prove the other containment, let $J' := ((J \cap R) : H)$. Then $BJ' = J'B$ since J' is H-stable hence K-stable by (1). Thus $J'BA = J'A = AJ' = ABJ' = AJ'B$, and $J'B = BJ'$ is an $\bar{H}$-stable ideal in B contained in J. Therefore $J' \subset (J : \bar{H}) \cap R$. ∎

We can now show transitivity for "lying over".

COROLLARY 6.4: *Let $P \in \operatorname{Spec}(A)$, $L \in \operatorname{Spec}(B)$, and $Q \in \operatorname{Spec}(R)$. If P lies over L and L lies over Q then P lies over Q.*

Proof: By assumption, $P \cap B = (L : \bar{H})$ and $L \cap R = (Q : K)$. Hence by 6.3,

$$\begin{aligned} P \cap R &= (P \cap B) \cap R = (L : \bar{H}) \cap R = ((L \cap R) : H) \\ &= ((Q : K) : H) = (Q : H). \quad ∎ \end{aligned}$$

We will prove our results on transitivity of the Krull relations in an axiomatic setting since we want to apply them in two different situations. Our abstract transitivity results will prove "going up" of the Krull relations from K and $\bar{H}$ to H. They will also be used in the next section to study the question of which Krull relations are reflected under field extensions.

Let $R \subset A$ be any ring extension; we wish to consider certain relations between prime ideals P in A and prime ideals Q in R. Formally, the relation is a set $\mathcal{R} \subset \operatorname{Spec}(A) \times \operatorname{Spec}(R)$. The relation is called a **lying over relation** if $P \cap R \subset Q$ for all $(P, Q) \in \mathcal{R}$. A lying over relation is called **strong** if $P \cap R = P' \cap R$ for all $P, P' \in \operatorname{Spec}(A)$ lying over the same prime ideal in R.

In this section we will use the lying over relation of 2.3 for H-Galois extensions. In the next section we are given a field extension $k \subset E$ and study the base field extension $S \subset S \otimes E$, S any k-algebra. In this example, $P \in \operatorname{Spec}(S \otimes E)$ is said to lie over $Q \in \operatorname{Spec}(S)$ if $P \cap S = Q$. Both are examples of strong lying over relations.

Definition 6.5: Let $R \subset A$ be a ring extension with a lying over relation.

(1) Let $t, n \geq 1$. The extension $R \subset A$ has *(t,n)-coLO* if for all $P \in \operatorname{Spec}(A)$ there are $Q_1, \dots Q_n \in \operatorname{Spec}(R)$ such that P lies over all Q_i and $(\bigcap_{i=1}^n Q_i)^t \subset P \cap R$.

(2) $R \subset A$ has *GU* if for all $Q_1 \subset Q_2$ in $\operatorname{Spec}(R)$ and $P_1 \in \operatorname{Spec}(A)$ lying over Q_1 there is a $P_2 \in \operatorname{Spec}(A)$ lying over Q_2 and containing P_1.

(3) $R \subset A$ has *INC* if for all $P_1 \subsetneq P_2$ in $\operatorname{Spec}(A)$, $P_1 \cap R \neq P_2 \cap R$.

(4) We say $R \subset A$ has *property* $(\mathcal{P})$ if for all $P \in \operatorname{Spec}(A)$, $(P \cap R)A$ is an ideal in A and $((P \cap R)A) \cap R = P \cap R$.

THEOREM 6.6: *Let $R \subset B$ and $B \subset A$ be ring extensions with lying over relations. Define a lying over relation for $R \subset A$ by transitivity: $P \in \operatorname{Spec}(A)$ lies over $Q \in \operatorname{Spec}(R)$ if there is an $L \in \operatorname{Spec}(B)$ such that P lies over L and L lies over Q. Let m, n, s, and t be positive integers.*

1. *Assume $R \subset B$ has (s, m)-coLO and $B \subset A$ has (t, n)-coLO. Then $R \subset A$ has (st, mn)-coLO.*
2. *Assume $R \subset B$ has (s, m)-coLO and GU, and $B \subset A$ has (t, n)-coLO and GU. Then $R \subset A$ has GU.*
3. *Assume $R \subset B$ satisfies $(\mathcal{P})$ and has INC, and $B \subset A$ has a strong lying over relation with (t, n)-coLO and INC. Then $R \subset A$ has INC.*

Proof: (1) Let $P \in \operatorname{Spec}(A)$. Since $B \subset A$ has (t, n)-coLO, there are $L_1, \dots L_n \in \operatorname{Spec}(B)$ such that P lies over all L_i and $(\bigcap_{i=1}^n L_i)^t \subset P \cap B$. Since $R \subset B$ has (s, m)-coLO, for all $L_i \in \operatorname{Spec}(B)$ there are $Q_{i_1}, \dots Q_{i_m} \in \operatorname{Spec}(R)$ such that L_i lies over all Q_{i_j} and $(\bigcap_{j=1}^m Q_{i_j})^s \subset L_i \cap R$. Then

$$\left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m} Q_{i_j}\right)^{st} \subset \left(\bigcap_{i=1}^{n}\left(\bigcap_{j=1}^{m} Q_{i_j}\right)^{s}\right)^{t} \subset \left(\bigcap_{i=1}^{n}(L_i \cap R)\right)^{t} \subset \left(\bigcap_{i=1}^{n} L_i^t\right) \cap R$$
$$\subset P \cap B \cap R = P \cap R.$$

Also P lies over all Q_{i_j} since P lies over L_i and L_i lies over Q_{i_j} for all i, j.

(2) Let $Q_2 \subset Q_1$ in $\operatorname{Spec}(R)$ and $P_2 \in \operatorname{Spec}(A)$ lying over Q_2. Since $B \subset A$ has (t, n)-coLO there are $L_1, \dots, L_n \in \operatorname{Spec}(B)$ such that P_2 lies over all L_i and $(\bigcap_{i=1}^n L_i)^t \subset P_2 \cap B$. Then $(\bigcap_{i=1}^n (L_i \cap R))^t \subset (\bigcap_{i=1}^n L_i)^t \cap R \subset P_2 \cap R$. Since P_2 lies over Q_2, $P_2 \cap R \subset Q_2 \subset Q_1$. Hence $L_i \cap R \subset Q_1$ for some i since Q_1 is prime. Since $R \subset B$ has (s, m)-coLO there are $Q'_1, \dots Q'_m \in \operatorname{Spec}(R)$ such that L_i lies over all Q'_j and $(\bigcap_{j=1}^m Q'_j)^s \subset L_i \cap R$. Hence $Q'_j \subset Q_1$ for some j since $L_i \cap R$ is contained in the prime Q_1. Let $Q_0 := Q'_j$. Then $Q_0 \subset Q_1$ in $\operatorname{Spec}(R)$ and $L_i \in \operatorname{Spec}(B)$ lies over Q_0. By GU for $R \subset B$, there is a prime ideal L'_i in B

lying over Q_1 and containing L_i. By construction, P_2 lies over L_i. Hence by GU for $B \subset A$, there is a prime ideal P_1 in A lying over L_i' and containing P_2. This proves GU for $R \subset A$ since P_1 lies over L_i' and L_i' lies over Q_1.

(3) Let $P_2 \subsetneq P_1$ be prime ideals in A and assume $P_2 \cap R = P_1 \cap R$. Since $B \subset A$ has (t,n)-coLO there are $Q_1, \ldots, Q_n \in \operatorname{Spec}(B)$ such that P_1 lies over all Q_i and $(\bigcap_{i=1}^n Q_i)^t \subset P_1 \cap B$, and also $Q_1', \ldots, Q_n' \in \operatorname{Spec}(B)$ such that P_2 lies over all Q_j' and $(\bigcap_{j=1}^n Q_j')^t \subset P_2 \cap B$. Hence $(\bigcap_{j=1}^n Q_j')^t \subset P_2 \cap B \subset P_1 \cap B$. Since P_1 lies over Q_i, $P_1 \cap B \subset Q_i$ for all i. Thus for all i, $Q_j' \subset Q_i$ for some j. By INC for $B \subset A$ we know $P_2 \cap B \subsetneq P_1 \cap B$. Since $B \subset A$ has a strong lying over relation, P_2 and P_1 do not both lie over the same prime ideal in B. In particular, $Q_j' \neq Q_i$ for all i, j. By INC for $R \subset B$ we therefore have shown that for all i, $Q_j' \cap R \subsetneq Q_i \cap R$ for some j. Since $P_2 \cap R = P_1 \cap R$ by assumption,

$$(\bigcap_{i=1}^{n}(Q_i \cap R))^t \subset (\bigcap_{i=1}^{n} Q_i)^t \cap R \subset P_2 \cap R = P_1 \cap R.$$

Since P_2 lies over all Q_j', we have $P_2 \cap R = P_2 \cap B \cap R \subset Q_j' \cap R$. Therefore $(\bigcap_{i=1}^n (Q_i \cap R))^t \subset Q_j' \cap R$ for all j. Since $R \subset B$ satisfies property $(\mathcal{P})$, for all j, $(Q_l \cap R)B \subset Q_j'$ for some l and

$$Q_l \cap R = ((Q_l \cap R)B) \cap R) \subset Q_j' \cap R.$$

Altogether we have shown that for all i, $Q_l \cap R \subsetneq Q_i \cap R$ for some l. But this is impossible since the number of the Q_i is finite. ∎

We now go back to the situation of Galois extensions $R \subset B \subset A$ described in the beginning of this section. Theorem 6.6 together with 6.4 then implies

THEOREM 6.7: *Let H be a finite-dimensional Hopf algebra , K a normal sub Hopf algebra of H and $\bar{H} := H/HK^+$.*

1. *Assume K has s-LO (resp. s-coLO) and $\bar{H}$ has t-LO (resp. t-coLO). Then H has st-LO (resp. st-coLO).*
2. *Assume K has s-coLO and $\bar{H}$ has t-coLO (resp. s-LO and t-LO) for some s and t. If K and $\bar{H}$ have GU (resp. coGU), then so has H.*
3. *Assume $\bar{H}$ has t-coLO (resp. K has t-LO) for some t. If K and $\bar{H}$ have INC (resp. coINC), then so has H.*

Proof: In all cases, by 6.1 and 4.3 it suffices to prove the coLO version. Lying over as defined in 2.3(1) is clearly a strong lying over relation. By 1.3(1) and (2) faithfully flat Galois extensions satisfy property $(\mathcal{P})$. Hence the theorem follows from 6.6, since by 6.4 "lying over" for $R \subset A$ in the sense of 6.6 implies lying over. ∎

7. Extending the ground field

Let H be a finite-dimensional Hopf algebra . We first note that trivially all Krull relations for H are preserved under field extensions.

LEMMA 7.1: *Let $k \subset E$ be any field extension. If $R \subset A$ is a faithfully flat $H \otimes E$-Galois extension over the ground field E, then $R \subset A$ is also faithfully flat H-Galois over k by restriction of the ground field. In particular, if H satisfies one of the Krull relations, then so does $H \otimes E$.*

Proof: $R \subset A$ is an extension of E-algebras, hence of k-algebras by restriction. The $H \otimes E$-comodule structure of A defines an H-comodule algebra structure over k by $A \to A \otimes_E (H \otimes E) \cong A \otimes H$. The Galois map for A over k is

$$A \otimes_R A \to A \otimes_E (H \otimes E) \cong A \otimes H.$$

Hence $R \subset A$ is faithfully flat H-Galois. Note that t-LO, INC, ... are all defined in terms of H-Galois extensions $R \subset A$ and they only depend on the ring extension $R \subset A$. Hence the lemma follows trivially. ∎

The more difficult question is which properties of H are reflected under field extensions. Let $k \subset E$ be any field extension. If S is any k-algebra we will use "lying over" for the ring extension $S \subset S' = S \otimes E$ in the usual sense: $P' \in \operatorname{Spec}(S')$ lies over $P \in \operatorname{Spec}(S)$ if $P' \cap S = P$. For any ideal J in S', $(J \cap S)S' = (J \cap S)E$ is an ideal in S', and for any ideal I in S, $(IS') \cap S = I$. In particular, $S \subset S'$ satisfies property $(\mathcal{P})$ in the last section. The ring extension $S \subset S'$ has the following Krull relations:

LEMMA 7.2:
(1) $\operatorname{Spec}(S') \to \operatorname{Spec}(S)$, $P' \mapsto P' \cap S$, *is well-defined and surjective.*
(2) $S \subset S'$ *has GU.*
(3) *If $k \subset E$ is algebraic, then $S \subset S'$ has INC.*

Proof: (1), (2), and (3) are shown in [R, 2.12.39, 2.12.50, and 3.4.13′]. ∎

To apply the abstract transitivity results of the last section, we need the following cutting down lemma.

LEMMA 7.3: *Let H be a finite-dimensional Hopf algebra, $R \subset A$ a faithfully flat H-Galois extension and $k \subset E$ any field extension. Then $R' := R \otimes E \subset A' := A \otimes E$ is faithfully flat $H' = H \otimes E$-Galois over E.*

Assume $P' \in \operatorname{Spec}(A')$ lies over $Q' \in \operatorname{Spec}(R')$. Define $P := P' \cap A$ and $Q := Q' \cap R$. Then $P \in \operatorname{Spec}(A)$ lies over $Q \in \operatorname{Spec}(R)$.

Proof: (1) Trivially, $R' \subset A'$ is again faithfully flat H'-Galois. First we show for any ideal I in R and $I' := I \otimes E$, $(I' : H') \cap R = ((I' \cap R) : H)$.

To prove that $(I' : H') \cap R$ is contained in the right hand side, note that $((I' : H')A') \cap A$ is a sub H-comodule, hence is extended. Therefore,

$$((I' : H')A') \cap A = (((I' : H')A') \cap R)A = ((I' : H') \cap R)A.$$

Since $(I' : H')A' = A'(I' : H')$, by the same argument on the other side we get $((I' : H') \cap R)A = A((I' : H') \cap R)$. Thus $(I' : H') \cap R$ is H-stable, hence contained in $((I' \cap R) : H)$.

To prove the other inclusion, let J be any H-stable ideal of R contained in I'. Then $JA = AJ$ and $(JE)A' = A'(JE)$. Hence $J \subset JE \subset (I' : H')$.

(2) To prove the lemma, note that P and Q are prime ideals by 7.2(1). By assumption $P' \cap R' = (Q' : H')$. Hence

$$P \cap R = P' \cap R = (Q' : H') \cap R = (Q : H),$$

where the last equality follows from (1). ∎

THEOREM 7.4: *Let H be a finite-dimensional Hopf algebra and $k \subset E$ a field extension. Let t be a positive integer.*

1. *If $H \otimes E$ has t-LO (resp. t-coLO), then so does H.*
2. *Assume $H \otimes E$ has t-coLO and GU (resp. t-LO and coGU). Then H has GU (resp. coGU).*
3. *Assume $H \otimes E$ has t-coLO and INC (resp. t-LO and INC) and $k \subset E$ is algebraic. Then H has INC (resp. coINC).*

Proof: Let $R \subset A$ be faithfully flat H-Galois. Then $R' := R \otimes E \subset A' := A \otimes E$ is faithfully flat H'-Galois, where $H' := H \otimes E$. We consider "lying over" for the Galois extensions $R \subset A$ and $R' \subset A'$ as in 2.3(1) and for $R \subset R'$ and $A \subset A'$ as in 7.2. The idea in each case is to apply 6.6 for $R \subset R' \subset A'$. Cutting down by 7.3 then yields the desired result. Note that $R \subset R'$ has $(1,1)$-coLO, GU and INC by 7.2. By duality (4.3), it suffices to prove the unbracketed statements.

(1) Assume H' has t-LO. Let $P \in \operatorname{Spec}(A)$. By 7.2(1), $P' \cap A = P$ for some $P' \in \operatorname{Spec}(A')$. By 6.6(1), $R \subset A'$ has (t,n)-coLO, where $n = \dim H$. Hence

there are $Q_1, \dots, Q_n \in \mathrm{Spec}(R)$ such that P' lies over all Q_i and $(\bigcap_{i=1}^n Q_i)^t \subset P' \cap R = P \cap R$. Thus for all i, P' lies over some $Q_i' \in \mathrm{Spec}(R')$ with $Q_i' \cap R = Q_i$. Hence P lies over all Q_i by 7.3.

(2) Let $Q_2 \subset Q_1$ in $\mathrm{Spec}(R)$ and $P_2 \in \mathrm{Spec}(A)$ lying over Q_2. By 7.2(1), $P_2' \cap A = P_2$ for some $P_2' \in \mathrm{Spec}(A')$. By 6.6(2), $R \subset A'$ has GU and there is a prime P_1' in A' lying over Q_1 and containing P_2'. Thus P_1' lies over some $Q_1' \in \mathrm{Spec}(R')$ with $Q_1' \cap R = Q_1$. Then $P_1 := P_1' \cap A \supset P_2$ is in $\mathrm{Spec}(A)$ by 7.2(1) and lies over Q_1 by 7.3.

(3) Let $P_2 \subsetneq P_1$ in $\mathrm{Spec}(A)$. By 7.2(1) and (2), there are $P_2' \subsetneq P_1'$ in $\mathrm{Spec}(A')$ such that $P_2' \cap A = P_2$ and $P_1' \cap A = P_1$. Since $R \subset A'$ has INC by 6.6(3),

$$P_2 \cap R = P_2' \cap R \subsetneq P_1' \cap R = P_1 \cap R. \quad \blacksquare$$

8. Consequences

Let H be a finite-dimensional Hopf algebra . A **normal series**

$$H_{n+1} = k \subset H_n \subset \cdots \subset H_0 = H$$

is a sequence of sub Hopf algebras such that H_{i+1} is a normal sub Hopf algebra of H_i for all $0 \le i \le n$. The quotient Hopf algebras $\bar{H}_i := H_i/H_iH_{i+1}^+$ are the **quotients** of the normal series.

From 6.6 we immediately get

Corollary 8.1: *Suppose H has a normal series with quotients $\bar{H}_i$, $0 \le i \le n$. Let s_i, t_i for $0 \le i \le n$ be positive integers.*

(1) *Assume all $\bar{H}_i$ have s_i-coLO. Then H has s-coLO, where $s := s_0 s_1 \cdots s_n$, and if all quotients have INC (resp. GU), then so does H.*
(2) *Assume all $\bar{H}_i$ have t_i-LO. Then H has t-LO, where $t := t_0 t_1 \cdots t_n$, and if all quotients have coINC (resp. coGU), then so does H.*

We first want to derive consequences from 8.1 and 7.4 together with known results for pointed Hopf algebras in 4.10.

We call H **solvable** (resp. **cosolvable**) if H has a normal series with commutative (resp. cocommutative) quotients.

Lemma 8.2:

(1) *Let K be a sub Hopf algebra and L a normal sub Hopf algebra of H. Then the canonical map $K/K(K \cap L)^+ \to H/HL^+$ is injective. Thus the second isomorphism theorem for finite-dimensional Hopf algebras holds, that is,*

$$K/K(K \cap L)^+ \xrightarrow{\cong} (KL)/(KL)L^+.$$

(2) *Let K be a sub Hopf algebra and $\bar{H}$ a quotient Hopf algebra of H. If H is solvable (resp. cosolvable), then so are K and $\bar{H}$.*

Proof: (1) We first have to show that $K \cap HL^+ = K(K \cap L)^+$. Let $p\colon H \to H/HL^+$ be the canonical map. Consider H as a left $p(H)$-comodule algebra via $(p \otimes \mathrm{id})\Delta$. Then $L = {}^{\mathrm{co}\, p(H)}H$ by [T, Th. 1] since H is (left) faithfully flat over L by [NZ]. Hence $K \cap L = {}^{\mathrm{co}\, p(K)}K$. Again by [NZ], K is (right) faithfully coflat over $p(K)$ since K^* is free over $p(K)^*$. Hence we get from [T, Th. 2] that $K \cap HL^+ = K({}^{\mathrm{co}\, p(K)}K)^+$. Thus $K \cap HL^+ = K(K \cap L)^+$, and so $K/K(K\cap L)^+ \to H/HL+$ is injective. It then follows that the canonical map $K/K(K \cap L)^+ \to (KL)/(KL)L^+$ is injective; it is clearly surjective.

(2) Let $H_{n+1} = k \subset H_n \subset \cdots \subset H_0 = H$ be a normal series of H. Define $K_i := H_i \cap K$ for all i. Then

$$k = K_{n+1} \subset K_n \subset \cdots \subset K_1 \subset K_0 = K$$

is a normal series of K. For all i, the canonical map $\bar{K}_i := K_i/K_iK_{i+1}^+ \to H_i/H_iH_{i+1}^+ =: \bar{H}_i$ is injective by (1) applied to the sub Hopf algebra K_i and the normal sub Hopf algebra H_{i+1} of H_i.

Similarly, let $\pi\colon H \to \bar{H}$ be the canonical projection and define $\bar{H}_i := \pi(H_i)$ for all i. Then

$$\bar{H}_{n+1} = k \subset \bar{H}_n \subset \cdots \subset \bar{H}_0 = \bar{H}$$

is a normal series for $\bar{H}$, and for all i, the natural map from $H_i/H_iH_{i+1}^+$ to $\bar{H}_i/\bar{H}_i\bar{H}_{i+1}^+$ is surjective. Now the claim is obvious. ∎

THEOREM 8.3: *Let $t = \dim H$.*

(1) *If H is cosolvable, then H has t-coLO and GU.*

(2) *If H is solvable, then H has t-LO and coGU.*

(3) *If H is solvable and cosolvable, for example if H is solvable and cocommutative, then H has t-LO, t-coLO, GU, coGU, INC and coINC.*

Proof: (1) Let K be one of the quotients of a normal series of H with cocommutative quotients. Then $K \otimes \bar{k}$, $\bar{k}$ an algebraic closure of k, is pointed. Hence, by 4.10 and 7.4, K has s-coLO and GU, where $s = \dim K$. By [M, 3.3.1], $t = \dim H$ is the product of the dimensions of all the quotients of the normal series. Then by 8.1(1), H has t-coLO and GU.

(2) This is shown as (1) using 4.3.

(3) Let $H_{n+1} = k \subset H_n \subset \cdots \subset H_0 = H$ be a normal series of H with cocommutative quotients $\bar{H}_i$. Since H is solvable, all $\bar{H}_i$ are solvable by 8.2.

Thus the quotients $\bar{H}_i$ have t-LO as seen in the proof of (2). By 4.10 and 7.4 they also have coINC and coGU. Therefore, H has coINC and coGU by 8.1(2). Starting with a normal series with commutative quotients, it follows similarly that H has INC. ∎

Remark 8.4: Let $u(\mathfrak{g})$ be the restricted universal enveloping algebra of a finite-dimensional restricted Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is abelian, then Chin [Ch87, Th. 19] showed that any crossed product extension $R \subset R\#_\sigma u(\mathfrak{g})$ satisfies INC. By 8.3(3), the same result holds for solvable Lie algebras $\mathfrak{g}$. In fact, $u(\mathfrak{g})$ has all the Krull relations for solvable $\mathfrak{g}$ by 8.3(3).

The question as to whether cocommutative Hopf algebras have all the Krull relations can be reduced to the (open) case of restricted Lie algebras. This follows from 8.1 and 7.4, since over an algebraically closed field in positive characteristic any cocommutative Hopf algebra has an irreducible normal sub Hopf algebra H^1 with quotient being a group algebra, and any irreducible cocommutative Hopf algebra has a normal series with quotients of the form $u(\mathfrak{g})$, $\mathfrak{g}$ a restricted Lie algebra [Ga].

Theorem 8.3 can be greatly improved for semisimple Hopf algebras.

We call H **semisolvable** if H has a normal series with commutative or cocommutative quotients. The next result improves [MS, Theorem 7.12].

THEOREM 8.5: *Let H be semisimple and semisolvable of dimension t. Then H has 1-LO, t-coLO, GU, coGU, INC, coINC. In particular if $R \subset A$ is faithfully flat H-Galois and R is H-prime, for example if R is an H-prime H-module algebra and $A = R\#H$, then*

(1) *A has at most $n \leq \dim H$ minimal primes, call them $P_1, \ldots, P_n$;*
(2) *P in $\operatorname{Spec}(A)$ is minimal if and only if $P \cap R = \{0\}$;*
(3) $\bigcap_i P_i = \{0\}$.

Proof: Note that (1)–(3) follow from 1-LO and INC.

We first consider the case when $H = kG$. Then H has all the Krull relations, as discussed in Example 4.9; in fact, H has 1-coLO. Since H is semisimple, H also has 1-LO. This follows from 4.5(1) and 4.6, since in the semisimple case $R\#H$ is semiprime for any H-prime H-module algebra R by [FM]. By dualizing 4.9 we see that $(kG)^*$ has all the Krull relations and satisfies 1-LO.

Now say H is commutative. By extending the base field and using Theorem 7.4, we may assume that $H = (kG)^*$ (we only need an algebraic extension for this), and thus have the desired conclusion.

If H is cocommutative, then H is pointed after a finite field extension of the base field. Thus we may assume that $H = K\#kG$, where $K = (kL)^*$ for some groups L, G (using some classical structure theorems as we did in [MS, 7.12]). But now K is normal in H, so we may use transitivity to get all the desired Krull relations for H.

Now use transitivity on any appropiate normal series with commutative or cocommutative quotients. ∎

THEOREM 8.6: *Let H be cosemisimple and semisolvable of dimension t. Then H has t-LO, 1-coLO, GU, coGU, Inc, coINC.*

Proof: As in the previous proof it suffices to show the result for commutative and cocommutative Hopf algebras H. If H is commutative (resp. cocommutative) and cosemisimple, then H^* is cocommutative (resp. commutative) and semisimple. Hence H has the required Krull relations by 8.5 (for H^*) and 4.3. ∎

COROLLARY 8.7: *Assume one of the following:*

(1) *H is semisimple, cosemisimple and semisolvable.*

(2) *H is semisimple, the characteristic of k is 0 and the dimension of H is a power of a prime.*

Then H has 1-LO, 1-coLO, GU, coGU, INC, coINC.

Proof: In case (1) this follows from 8.5 and 8.6. Assume (2). By 7.4 it suffices to consider the case when k is algebraically closed. Then H contains a non-trivial central group-like element by Masuoka and Zhu [Ma96]. Hence H contains a non-trivial normal sub Hopf algebra K which is a (commutative) semisimple group algebra and therefore satisfies the required Krull relations by (1) (or by 4.9, since K and $K^* \cong K$ are group algebras). Alternatively, one can show that (2) is a special case of (1) by applying [Ma96] to H^*. ∎

We now specialize to the semiprimeness problem raised in [CF]. Let H be a semisimple Hopf algebra and R a left H-module algebra which is H-prime . Then the question is whether the smash product $R\#H$ is semiprime.

Definition 8.8: (1) H is called **strongly semisimple,** if for all left H-module algebras A with ring of invariants $R := A^H$ and for all $P \in \mathrm{Spec}(A)$, $P \cap R$ is a semiprime ideal in R.

(2) H is called **strongly cosemisimple**, if for all right H-comodule algebras A with ring of coinvariants $R := A^{\mathrm{co}H}$ and for all $P \in \mathrm{Spec}(A)$, $P \cap R$ is a semiprime ideal in R.

Remark 8.9: (1) H is strongly cosemisimple if and only if H^* is strongly semisimple.

(2) If H is strongly semisimple (resp. strongly cosemisimple), then H is semisimple (resp. cosemisimple).

(3) Any sub Hopf algebra of a strongly cosemisimple Hopf algebra is strongly cosemisimple.

(4) Any quotient Hopf algebra of a strongly semisimple Hopf algebra is strongly semisimple.

Proof: (1) is clear from the definition.

(2) Let H be strongly semisimple. Then H is a left H^*-module algebra and $H\#H^*$ is a right H^*-comodule algebra with H^*-coinvariants $R = H$, hence a left H-module algebra. Moreover, $H\#H^* \cong M_n(k)$, where $n = \dim H$, is prime. Hence H is semiprime by assumption. Thus H is semisimple.

(3) Let K be a sub Hopf algebra of H. Then any right K-comodule algebra A is also a right H-comodule algebra with the same ring of coinvariants R. Hence if H is cosemisimple, $P \cap R$ is semiprime for all $P \in \operatorname{Spec}(A)$.

(4) is dual to (3). ∎

THEOREM 8.10: *The following are equivalent:*

(1) *H is strongly semisimple.*

(2) *For all faithfully flat H-Galois extensions $R \subset A$ with R being H-prime, A is semiprime.*

(3) *For all left H-module algebras R which are H-prime , $R\#H$ is semiprime.*

Proof: (1) $\Rightarrow$ (2): Let $R \subset A$ be faithfully flat H-Galois and R be H-prime. Then by 4.2, $A\#H^*$ is an H-prime left H-module algebra with $(A\#H^*)^H = A$. Hence $P \cap A = \{0\}$ for some $P \in \operatorname{Spec}(A\#H^*)$ by 2.2(3). Since H is strongly semisimple, $\{0\}$ is a semiprime ideal, and A is semiprime.

(2)$\Rightarrow$(3) holds trivially.

(3)$\Rightarrow$(1) Let A be any left H-module algebra and $R := A^H$. By (3), H is semisimple (apply (3) to the trivial H-module algebra k). Hence Section 5 applies. By 5.4(1), $\varphi((P : H)\#H) = P \cap R$. We will show below that $(P : H)\#H$ is a semiprime ideal in $A\#H$. Hence $(P : H)\#H$ is an intersection of prime ideals in $A\#H$. Since φ in 5.4 preserves intersections, $P \cap R = \varphi((P : H)\#H)$ is an intersection of φ-images of prime ideals in R. Then $P \cap R$ is an intersection of prime ideals, since for any prime ideal L of $A\#H$, $\varphi(L)$ is a prime ideal if $e \notin L$ by 5.2 or $\varphi(L) = R$ if $e \in L$.

It remains to show that for any $I \in H\text{-Spec}(A)$ (such as $(P : H)$), $I\#H$ is a semiprime ideal in $A\#H$. To prove this, let $\bar{A} := A/I$. Then $\bar{A}$ is an H-prime left H-module algebra. By assumption (3), $\bar{A}\#H$ is semiprime. Hence $\{0\}$ in $\bar{A}\#H$ is an intersection of prime ideals in $\bar{A}\#H$, and $I\#H$ is the intersection of their inverse images in $A\#H$. Thus $I\#H$ is semiprime. ∎

THEOREM 8.11: *The following are equivalent:*

(1) *H is strongly cosemisimple.*

(2) *For all faithfully flat H-Galois extensions $R \subset A$ with R being H-prime , R is semiprime.*

(3) *For all left H-module algebras R which are H-prime , R is semiprime.*

Proof: This follows by duality from 8.9.

(1) ⇒ (2): By 8.9, H^* is strongly semisimple. Hence H^* satisfies condition (3) in 8.10. To show (2), let $R \subset A$ be faithfully flat H-Galois with R being H-prime. Then A is an H^*-prime left H^*-module algebra by 2.2(1). By 8.10(3) for H^*, $A\#H^*$ is semiprime. Hence R is semiprime by 1.4.

(2) ⇒ (3) is trivial (look at the H-Galois extension $R \subset R\#H$).

(3) ⇒ (1): By 8.10, it suffices to verify condition (2) in 8.10 for H^*. Thus let $R \subset A$ be a faithfully flat H^*-Galois extension with R being H^*-prime. By 2.2(1) again, A is an H-prime module algebra. Hence A is semiprime by assumption (3). ∎

We now introduce H-semiprime ideals in order to prove H-semiprime versions of the two previous theorems.

Definition 8.12: Let $R \subset A$ be faithfully flat H-Galois. An H-stable ideal N of R is H-**semiprime** if for any H-stable ideal I of R, $I^2 \subset N$ implies $I \subset N$. R itself is called H-semiprime if $\{0\}$ is an H-semiprime ideal.

Note that 8.12 defines as a special case the usual notion of an H-semiprime ideal in any left H-module algebra R (where $A = R\#H$).

The classical Levitzki–Nagata argument can be extended to get an analog of the usual characterization of semiprime ideals:

LEMMA 8.13: *Assume that H is finite-dimensional. Then an H-stable ideal N of R is H-semiprime if and only if $N = \bigcap P$, the intersection of all $P \in H\text{-Spec}(R)$ with $P \supset N$.*

Proof: By Lemma 1.6 and the remark which follows it, we may assume that $N = \{0\}$; that is, R is H-semiprime. Let $I = \bigcap P$, the intersection of all $P \in H\text{-Spec}(R)$; we claim $I = \{0\}$.

If not, then since I is H-stable , there exists a finitely generated H-stable ideal I_1 with $\{0\} \neq I_1 \subset I$, by Corollary 1.5. Now $I_1^2 \neq \{0\}$ since R is H-semiprime, and thus again by Corollary 1.5 we may choose a finitely generated H-stable ideal $0 \neq I_2 \subset I_1^2$. Continuing, we obtain a decreasing chain of finitely generated H-stable ideals $\{I_n\}$ such that $0 \neq I_n \subset (I_{n-1})^2$ for all $n > 0$.

We claim that we may apply Zorn's lemma to the set

$$\mathcal{S} = \{J \lhd R \mid J\ H\text{-stable},\ I_n \not\subset J, \text{ for all } n \geq 1\}.$$

For, $\mathcal{S}$ is non-empty since $\{0\} \in \mathcal{S}$, and $\mathcal{S}$ is closed under ascending unions since all the I_n are finitely generated. Thus we may choose P maximal in $\mathcal{S}$.

We claim that P is H-prime . For if $LM \subset P$ with $L \supsetneq P$ and $M \supsetneq P$, for L, M H-stable ideals of R, then $I_n \subset L$ and $I_m \subset M$ for some n, m by the maximality of P. Say $m \geq n$; then $I_{m+1} \subset (I_m)^2 \subset I_m I_n \subset LM \subset P$, a contradiction. Thus P is H-prime . But $I \not\subset P$, a contradiction. Thus $I = \{0\}$. ∎

Corollary 8.14: *In 8.10 and 8.11,* (2) *(resp.* (3)*) is equivalent to* (2)′ *(resp.* (3)′*) where the condition H-prime is replaced by H-semiprime.*

Proof: This follows in a standard way from 8.13 using 1.6 and 1.7. ∎

We finally note that our previous results about the Krull relations for semi-solvable Hopf algebras give a large class of examples of strongly semisimple or cosemisimple Hopf algebras.

Remark 8.15: If H has 1-LO (resp. 1-coLO), then H is strongly semisimple (resp. strongly cosemisimple).

Proof: (1) Recall from 4.5 that H has 1-LO if and only if for all faithfully flat Hopf Galois extensions $R \subset A$ with R being H-prime there exist $P_1, \ldots, P_n \in \mathrm{Spec}(R)$ with $n \leq \dim(H)$ such that $\bigcap_{i=1}^n P_i = \{0\}$ and $P_i \cap R = \{0\}$ for all i. In particular A then is semiprime. Hence if H has 1-LO then H is strongly semisimple by 8.10(2).

(2) If H has 1-coLO, then H^* has 1-LO by 4.3, hence H^* is strongly semisimple by (1), and H is strongly cosemisimple by 8.9. ∎

Corollary 8.16:

1. *Let H be semisimple (resp. cosemisimple) and semisolvable. Then H is strongly semisimple (resp. strongly cosemisimple).*
2. *Let H be semisimple and assume that the characteristic of k is 0 and the dimension of H is a power of a prime. Then H is strongly semisimple and strongly cosemisimple.*

Proof: Using the previous lemma, (1) follows from 8.5 and 8.6 and (2) follows from 8.7. ∎

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